ADVANCES IN MATHEMATICS 111, 314–354 (1995)

Character Sheaves and Almost Characters of Reductive Groups, II

TOSHIAKI SHOJI

Department of Mathematics, Science University of Tokyo, Noda, Chiba 278, Japan

CONTENTS

INTRODUCTION

This paper is a continuation of [S3]. In [S3], Lusztig's conjecture, which asserts the coincidence of almost characters of $G(\mathbf{F}_q)$ with characteristic functions of character sheaves, was proved for reductive groups G with connected center of type A_n, B_n, C_n, F_4, or G_2. In this paper, we treat the remaining cases, i.e., the case of type D_n, E_6, E_7, or E_8. Our strategy is fundamentally similar to [S3]. In the case of classical groups, first we show the conjecture assuming that q is large enough, and then extend it to arbitrary q by using a specialization argument. In the case of exceptional groups, we appeal to the property of twisting operators as used in [S3].

However, in [S3], in the case of classical groups the specialization was related to the twisted induction from the character sheaves of twisted type on a Levi subgroup (i.e., the character sheaves of disconnected groups). Contrast to it we develop in this paper a specialization associated to a different kind of twisted induction. The author was inspired the use of it from Lusztig [L4, 4]. It is a variant of the twisted induction in [S3, Section 4], and actually is an induction from the character sheaves of non-twisted type. So it has an advantage that one can avoid to use character sheaves of disconnected groups, (though the twisted induction in the sense of [S3] is also used in one step of the proof.) This simplifies the argument in[S3] for classical groups of type B_n or C_n. We prove, in Theorem 3.2, Lusztig's

0001-8708/95 $9.00

conjecture for classical groups in a uniform way. Lusztig's conjecture for exceptional groups is proved in Theorem 4.1.

In Section 5, we have included some results on character sheaves in the case of bad characteristic. In particular, we show in Theorem 5.5 that two kinds of Green functions, one is associated to Deligne–Lusztig's virtual characters, and the other is associated to character sheaves, coincide with each other without any restriction on the characteristic of $\mathbf{F}_q$ nor q.

1. A Variant of Twisted Induction

1.1. We follow the notation in [S3]. So, G will denote a connected reductive group defined over a finite field $\mathbf{F}_q$, with Frobenius map $F\colon G \to G$. Forgetting the $\mathbf{F}_q$-structure for a while, we consider the group G over k, where k is an algebraic closure of $\mathbf{F}_q$. We fix a Borel subgroup B of G, a maximal torus T contained in B and a Weyl group $W = N_G(T)/T$ of G with respect to T. Let P be a parabolic subgroup of G containing B, L the Levi subgroup of P containing T, and U_P the unipotent radical of P. Then $B_L = B \cap L$ is a Borel subgroup of L containing T. Let $W_L = N_L(T)/T \subset W$ be a Weyl group of L. In [S3, Sect. 4], the twisted induction $w\text{-ind}_P^G$ of character sheaves was defined for each $w \in N_W(W_L)$ such that $wB_Lw^{-1} = B_L$. In this section we consider some variant of it. Let $\mathbf{w} = (w_0, w_1, ..., w_{r-1})$ be a sequence in W such that $w_i \in N_W(W_L)$ and that $w_iB_Lw_i^{-1} = B_L$ for each i. We assume that

$$w_0 w_1 w_2 \cdots w_{r-1} = 1 \qquad \text{in } W. \tag{1.1.1}$$

Consider the diagram

$$L \xleftarrow{\rho} \hat{V} \xrightarrow{\pi'} V \xrightarrow{\pi} G,$$

where

$$\begin{aligned} V = \{&(g, h_0P, h_1P, ..., h_rP) \in G \times G/P \times G/P \times \cdots \times G/P \\ &\mid h_i^{-1}h_{i+1} \in Pw_iP\ (0 \leqslant i \leqslant r-1),\ h_r^{-1}gh_0 \in P\}, \\ \hat{V} = \{&(g, h_0U_P, h_1P, ..., h_{r-1}P, h_rU_P) \in G \times G/U_P \times G/P \times \cdots \times G/P \times G/U_P \\ &\mid h_i^{-1}h_{i+1} \in Pw_iP\ (0 \leqslant i \leqslant r-1),\ h_r^{-1}gh_0 \in U_P\}, \end{aligned}$$

and

$$\pi(g, h_0P, h_1P, ..., h_rP) = g,$$

$$\pi'(g, h_0U_P, h_1P, ..., h_{r-1}P, h_rU_P) = (g, h_0P, h_1P, ..., h_{r-1}P, h_rP).$$

The map $p\colon \hat{V} \to L$ is defined by

$$p(g, h_0 U_P, h_1 P, \dots, h_{r-1} P, h_r U_P) = \beta_0(h_0^{-1} h_1)\, \beta_1(h_1^{-1} h_2) \cdots \beta_{r-1}(h_{r-1}^{-1} h_r),$$

where $\beta_i\colon P w_i P \to w_i L$ is the projection to the $w_i L$-component of $P w_i P = U_P w_i L U_P$. We note that p is well-defined; in fact, the image of p is contained in $w_0 L \cdot w_1 L \cdots w_{r-1} L = L$ by the assumption (1.1.1), and it is easily checked that p does not depend on the choice of representatives $h_0, h_1, \dots, h_r$. The maps p and π' are smooth morphisms with connected fibres. Moreover, by the assumption (1.1.1), p is L-equivariant for the action of L on $\hat{V}$;

$$\begin{aligned} l\colon &(g, h_0 U_P, h_1 P, \dots, h_{r-1} P, h_r U_P) \\ &\mapsto (g, h_0 l^{-1} U_P, h_1 P, \dots, h_{r-1} P, h_r l^{-1} U_P), \end{aligned}$$

and for the conjugation action of L on L.

We shall now define a twisted induction $\mathbf{w}\text{-ind}_P^G K$ for each L-equivariant perverse sheaf K on L (with respect to the conjugation action of L on L). Let K be an L-equivariant perverse sheaf on L. Then $\tilde{p}K \in \mathcal{M}\hat{V}$ is L-equivariant, and since π' is a locally trivial principal L-bundle, there exists a well-defined perverse sheaf $K_1 \in \mathcal{M}V$ such that $\tilde{p}K = \tilde{\pi}' K_1$. We define $\mathbf{w}\text{-ind}_P^G K = \pi_! K_1$. We remark that this type of induction already appears in Lusztig [L4, 4] in a more general form. We also note that in the special case where $P = B$, this is defined in [L3, I, 2.5] without the assumption of (1.1.1), (see also [S3, Sect. 2]).

1.2. We shall describe here the variety $\hat{V}$ and the map p in a more explicit way, which will be convenient for our later discussion. Let $\mathbf{w} = (w_0, \dots, w_{r-1})$ be as in 1.1. We choose a representative $\dot{w}_i \in N_G(L)$ for each w_i and put $\dot{\mathbf{w}} = (\dot{w}_0, \dots, \dot{w}_{r-1})$. Consider the variety

$$\begin{aligned} \hat{V}' = \{ &(g, h_0 U_P, h_1 U_P, \dots, h_{r-1} U_P) \in G \times G/U_P \times G/U_P \times \cdots \times G/U_P \\ &\mid h_i^{-1} h_{i+1} \in U_P \dot{w}_i U_P \ (0 \leqslant i \leqslant r-2),\ h_{r-1}^{-1} g h_0 \in U_P \dot{w}_{r-1} P \}. \end{aligned}$$

We have a natural map $\hat{V}' \to \hat{V}$ by

$$(g, h_0 U_P, h_1 U_P, \dots, h_{r-1} U_P) \mapsto (g, h_0 U_P, h_1 P, \dots, h_{r-1} P, g h_0 U_P).$$

This map gives rise to an isomorphism $\hat{V}' \xrightarrow{\sim} \hat{V}$. In fact the inverse map is constructed as follows. Take $(g, h_0 U_P, h_1 P, \dots, h_{r-1} P, h_r U_P) \in \hat{V}$. Since $h_0^{-1} h_1 \in P w_0 P = U_P \dot{w}_0 L U_P$, there exists a unique $h_1' U_P$ such that $h_0^{-1} h_1' \in U_P \dot{w}_0 U_P$ and that $h_1' P = h_1 P$. Repeating this, one can find $h_1' U_P, \dots, h_{r-1}' U_P$ such that $(h_i')^{-1} h_{i+1}' \in U_P \dot{w}_i U_P$ $(1 \leqslant i \leqslant r-2)$ and that

$h'_iP = h_iP$ for $i = 1, ..., r-1$. Then $(h'_{r-1})^{-1}gh_0 \in P\dot{w}_{r-1}P = U_P\dot{w}_{r-1}P$, and the map

$$(g, h_0U_P, h_1P, ..., h_{r-1}P, h_rU_P) \mapsto (g, h_0U_P, h'_1U_P, ..., h'_{r-1}U_P)$$

gives the inverse map $\hat{V} \to \hat{V}'$. Note that, under this isomorphism the map $\pi' : \hat{V} \to V$ is transfered to a map

$$\pi'' : \hat{V}' \to V,$$
$$(g, h_0U_P, h_1U_P, ..., h_{r-1}U_P) \mapsto (g, h_0P, h_1P, ..., h_{r-1}P, gh_0P),$$

and the map $\rho : \hat{V} \to L$ is transfered to a map $\rho' : \hat{V}' \to L$, $(g, h_0U_P, ..., h_{r-1}U_P) \mapsto \dot{w}_0\dot{w}_1 \cdots \dot{w}_{r-1}l$, where $l \in L$ is given by the condition that $h_{r-1}^{-1}gh_0 \in U_P\dot{w}_{r-1}lU_P$.

1.3. We shall investigate more closely the twisted induction for $P = B$ along the setup in [L3, III]. Let $\mathscr{L} \in \mathscr{S}(T)$ be a tame local system, and let $W'_{\mathscr{L}}$ be the stabilizer of $\mathscr{L}$ in W under the action $w : \mathscr{L} \mapsto w\mathscr{L} = (w^{-1})^* \mathscr{L}$. (Note that $W'_{\mathscr{L}}$ was denoted as $W_{\mathscr{L}}$ in [S3]). We fix a sequence $\mathbf{w} = (w_0, w_1, ..., w_{r-1})$ of W such that $w_0w_1 \cdots w_{r-1} \in W'_{\mathscr{L}}$. But we do not assume (1.1.1) for $\mathbf{w}$. We fix $\dot{\mathbf{w}}$ by choosing $\dot{w}_i \in N_G(T)$ for each $w_i \in W$. Consider the diagram,

$$\begin{tikzcd}
 & \bar{X}_{\dot{\mathbf{w}}} \arrow[r, "\bar{\beta}_{\mathbf{w}}"] & \bar{Y}_{\mathbf{w}} \arrow[dr, "\bar{\pi}_{\mathbf{w}}"] & \\
T & X_{\dot{\mathbf{w}}} \arrow[l, "\alpha_{\dot{\mathbf{w}}}"'] \arrow[u, hook] \arrow[r, "\beta_{\mathbf{w}}"] & Y_{\mathbf{w}} \arrow[u, hook] \arrow[r, "\pi_{\mathbf{w}}"] & G,
\end{tikzcd}$$

where $\pi_{\mathbf{w}} : Y_{\mathbf{w}} \to G$ is similar to the map $\pi : V \to G$ in 1.1, namely,

$$Y_{\mathbf{w}} = \{(g, h_0B, h_1B, ..., h_rB) \in G \times G/B \times G/B \times \cdots \times G/B$$
$$\mid h_i^{-1}h_{i+1} \in Bw_iB \ (0 \leqslant i \leqslant r-1), h_r^{-1}gh_0 \in B\},$$

and

$$\pi_{\mathbf{w}}(g, h_0B, h_1B, ..., h_rB) = g.$$

Moreover

$$\bar{Y}_{\mathbf{w}} = \{(g, h_0B, h_1B, ..., h_rB) \in G \times G/B \times G/B \times \cdots \times G/B$$
$$\mid h_i^{-1}h_{i+1} \in \overline{Bw_iB} \ (0 \leqslant i \leqslant r-1), h_r^{-1}gh_0 \in B\},$$
$$X_{\dot{\mathbf{w}}} = \{(g, h_0, h_1, ..., h_r) \in G \times G \times \cdots \times G$$
$$\mid h_i^{-1}h_{i+1} \in B\dot{w}_iB \ (0 \leqslant i \leqslant r-1), h_r^{-1}gh_0 \in B\},$$
$$\bar{X}_{\dot{\mathbf{w}}} = \{(g, h_0, h_1, ..., h_r) \in G \times G \times \cdots \times G$$
$$\mid h_i^{-1}h_{i+1} \in \overline{B\dot{w}_iB} \ (0 \leqslant i \leqslant r-1), h_r^{-1}gh_0 \in B\},$$

and

$$\beta_{\mathbf{w}}(g, h_0, h_1, \ldots, h_r) = (g, h_0 B, h_1 B, \ldots, h_r B),$$

$$\alpha_{\dot{\mathbf{w}}}(g, h_0, h_1, \ldots, h_r) = (\dot{w}_0 \cdots \dot{w}_{r-1})^{-1}$$

$$\times (\beta_0(h_0^{-1}h_1)\, \beta_1(h_1^{-1}h_2) \cdots \beta_{r-1}(h_{r-1}^{-1}h_r)),$$

where $\beta_i \colon Bw_iB \to w_iT$ is the projection. Note that

$$\beta_0(h_0^{-1}h_1)\, \beta_1(h_1^{-1}h_2) \cdots \beta_{r-1}(h_{r-1}^{-1}h_r) \in \dot{w}_0\dot{w}_1 \cdots \dot{w}_{r-1}T.$$

The map $\bar{\beta}_{\mathbf{w}} \colon \bar{X}_{\dot{\mathbf{w}}} \to \bar{Y}_{\mathbf{w}}$ (resp. $\bar{\pi}_{\mathbf{w}} \colon \bar{Y}_{\mathbf{w}} \to G$) is defined similar to $\beta_{\mathbf{w}}$ (resp. $\pi_{\mathbf{w}}$). The maps $X_{\dot{\mathbf{w}}} \subsetneq \bar{X}_{\dot{\mathbf{w}}}$, $Y_{\mathbf{w}} \subsetneq \bar{Y}_{\mathbf{w}}$ are natural imbeddings. Note that $\alpha_{\dot{\mathbf{w}}}$ and $\beta_{\mathbf{w}}$ are smooth morphisms with connected fibres. Let $\mathbf{B} = B \times B \times \cdots \times B$, ($r$-factors). Then $\alpha_{\dot{\mathbf{w}}}$ is $\mathbf{B}$-equivariant for the $\mathbf{B}$-action on $X_{\dot{\mathbf{w}}}$; for $\mathbf{b} = (b_0, b_1, \ldots, b_{r-1}) \in \mathbf{B}$,

$$\mathbf{b} \colon (g, h_0, h_1, \ldots, h_r) \mapsto (g, h_0 b_0^{-1}, h_1 b_1^{-1}, \ldots, h_{r-1} b_{r-1}^{-1}, h_r b_0^{-1}),$$

and the $\mathbf{B}$-action on T;

$$\mathbf{b} \colon t \mapsto (\dot{w}_0 \cdots \dot{w}_{r-1})^{-1}\, t_0 (\dot{w}_0 \cdots \dot{w}_{r-1})\, t t_0^{-1} \qquad \text{where} \quad b_0 = t_0 u_0 \in TU.$$

Put $\hat{\mathcal{L}}_{\dot{\mathbf{w}}} = \alpha_{\dot{\mathbf{w}}}^* \mathcal{L}$. Since $w_0 w_1 \cdots w_{r-1} \in W'_{\mathcal{L}}$, $\mathcal{L}$ is $\mathbf{B}$-equivariant with respect to this action of $\mathbf{B}$ on T. It follows that $\hat{\mathcal{L}}_{\dot{\mathbf{w}}}$ is a $\mathbf{B}$-equivariant local system on $X_{\dot{\mathbf{w}}}$. Since $\beta_{\mathbf{w}}$ is a principal $\mathbf{B}$-bundle, there is a canonical local system $\tilde{\mathcal{L}}_{\dot{\mathbf{w}}}$ on $Y_{\mathbf{w}}$ such that $\beta_{\mathbf{w}}^* \tilde{\mathcal{L}}_{\dot{\mathbf{w}}} = \hat{\mathcal{L}}_{\dot{\mathbf{w}}}$. Now $X_{\dot{\mathbf{w}}}$ is an open dense subvariety of $\bar{X}_{\dot{\mathbf{w}}}$, and $X_{\dot{\mathbf{w}}}$ is smooth and irreducible. So the similar property holds for $Y_{\mathbf{w}} \subsetneq \bar{Y}_{\mathbf{w}}$. Let $J_{\dot{\mathbf{w}}}^{\mathcal{L}} = \mathrm{IC}(\bar{Y}_{\mathbf{w}}, \tilde{\mathcal{L}}_{\dot{\mathbf{w}}})$, $\hat{J}_{\dot{\mathbf{w}}}^{\mathcal{L}} = \mathrm{IC}(\bar{X}_{\dot{\mathbf{w}}}, \hat{\mathcal{L}}_{\dot{\mathbf{w}}})$. Then $\hat{J}_{\dot{\mathbf{w}}}^{\mathcal{L}} = \bar{\beta}_{\dot{\mathbf{w}}}^* J_{\dot{\mathbf{w}}}^{\mathcal{L}}$. We define complexes $\bar{K}_{\dot{\mathbf{w}}}^{\mathcal{L}} = (\bar{\pi}_{\mathbf{w}})_! J_{\dot{\mathbf{w}}}^{\mathcal{L}}$, $K_{\dot{\mathbf{w}}}^{\mathcal{L}} = (\pi_{\mathbf{w}})_! \tilde{\mathcal{L}}_{\dot{\mathbf{w}}}$ in $\mathcal{D}G$. Note that $K_{\dot{\mathbf{w}}}^{\mathcal{L}}$ coincides with $K_{\mathbf{w}}^{\mathcal{L}}$ in [L3, I, 2.5].

1.4. We review here the construction of $\mathcal{L} \in \mathcal{S}(T)$ and the structure of $W'_{\mathcal{L}}$ given in [L3, I]. For an invertible integer $n \geqslant 1$ in k, let $\mu_n = \{x \in k^* \mid x^n = 1\}$. We consider the principal fibration $\rho_n \colon k^* \to k^*$, $x \mapsto x^n$ with group μ_n. We denote by $\mathcal{E}_{n,\psi}$ the irreducible summand of $(\rho_n)_* \bar{\mathbf{Q}}_l$ on which μ_n acts according to the character $\psi \colon \mu_n \to \bar{\mathbf{Q}}_l^*$. We fix an imbedding $\tilde{\psi} \colon \{\text{roots of 1 in } k^*\} \subsetneq \bar{\mathbf{Q}}_l^*$. Then any $\mathcal{L} \in \mathcal{S}(T)$ is expressed as $\mathcal{L} = \lambda^*(\mathcal{E}_{n,\psi})$ for some $\lambda \in \mathrm{Hom}(T, k^*)$ and some integer $n \geqslant 1$, invertible in k, with the restriction $\psi \colon \mu_n \to \bar{\mathbf{Q}}_l^*$ of $\tilde{\psi}$. This gives rise to a group isomorphism

$$\mathrm{Hom}(T, k^*) \otimes (\mathbf{Q}'/\mathbf{Z}) \xrightarrow{\sim} \mathcal{S}(T), \qquad \lambda \otimes \frac{1}{n} \mapsto \lambda^*(\mathcal{E}_{n,\psi}), \tag{1.4.1}$$

where $\mathbf{Q}' = \{m/n \in \mathbf{Q} \mid m, n \in \mathbf{Z}, n\text{: invertible in } k\}$. Assume $\mathscr{L} = \lambda^*(\mathscr{E}_{n,\psi})$. We have

$$W'_{\mathscr{L}} = \{w \in W \mid w(\lambda) = \lambda\lambda_1^n \text{ for some } \lambda_1 \in \operatorname{Hom}(T, k^*)\}, \qquad (1.4.2)$$

where W acts on $\operatorname{Hom}(T, k^*)$ by $w(\lambda)(t) = \lambda(w^{-1}(t))$. Let $R \subset \operatorname{Hom}(T, k^*)$ be the set of roots and we denote by r_α the reflection corresponding to $\alpha \in R$. We define $W_{\mathscr{L}}$ as the subgroup of W generated by $r_\alpha \in W'_{\mathscr{L}}$.

We now consider the $\mathbf{F}_q$-structure on G with Frobenius map F, and assume that B and T are F-stable. There exists an integer $d > 0$ such that T split over $\mathbf{F}_{q^d}$ and that n divides $q^d - 1$. Then $\mathscr{L}^{\otimes(q^d-1)} \simeq \bar{\mathbf{Q}}_l$, and $\mathscr{L}$ has a natural $\mathbf{F}_{q^d}$-structure $(F^d)^* \mathscr{L} \xrightarrow{\sim} \mathscr{L}$ such that the characteristic function on T^{F^d} is given by $t \mapsto \psi(\lambda(t)^{(q^d-1)/n})$. More generally, F acts on $\mathscr{S}(T)$ by $F\colon \mathscr{L} \mapsto F\mathscr{L} = F_* \mathscr{L}$ (or equivalently, $F^{-1}\colon \mathscr{L} \mapsto F^*\mathscr{L}$). This action is compatible with the action $F\colon (\lambda, n) \mapsto (F(\lambda), n)$ on $\operatorname{Hom}(T, k^*) \otimes (\mathbf{Q}'/\mathbf{Z})$ under the isomorphism (1.4.1), where $F(\lambda)$ is defined by $F(\lambda)(t) = \lambda(F^{-1}(t))$. Moreover, we have $FwF^{-1} = F(w)$ as an action on $\mathscr{S}(T)$. We define

$$Z'_{\mathscr{L}} = \{w \in W \mid Fw(\lambda) = \lambda\lambda_1^n \text{ for some } \lambda_1 \in \operatorname{Hom}(T, k^*)\}. \qquad (1.4.3)$$

Then $Z'_{\mathscr{L}}$ is invariant under the right multiplication by $W_{\mathscr{L}}$. By (1.4.1) $w \in W$ is in $Z'_{\mathscr{L}}$ if and only if $\mathscr{L}$ is fixed by $F\dot{w}$. We assume that $Z'_{\mathscr{L}}$ is nonempty and we choose a coset $Z_{\mathscr{L}} = zW_{\mathscr{L}}$ in $Z'_{\mathscr{L}}$. Then $F(Z_{\mathscr{L}}) = F(z)\, W_{F\mathscr{L}}$ is a coset of $W_{F\mathscr{L}}$ in $Z'_{F\mathscr{L}}$. We choose an integer $r > 0$ such that

(1.4.4) r is a multiple of d.

Then $F^r\mathscr{L} \simeq \mathscr{L}$, and so $F^r(W'_{\mathscr{L}}) = W'_{\mathscr{L}}$, $F^r(Z_{\mathscr{L}}) = Z_{\mathscr{L}}$. We define $Z_i \subset W$ by $Z_i = F^{i+1}(Z_{\mathscr{L}})$ for $i = 0, 1, \dots, r-1$, and put $\mathbf{Z}_{\mathscr{L}} = Z_0 \times Z_1 \times \cdots \times Z_{r-1}$. We have the following

(1.4.5) If $\mathbf{w} = (w_0, w_1, \dots, w_{r-1}) \in \mathbf{Z}_{\mathscr{L}}$, then $w_0 w_1 \cdots w_{r-1} \in W'_{\mathscr{L}}$.

In fact, if $w \in Z_{\mathscr{L}}$, we have $Fw\mathscr{L} \simeq \mathscr{L}$, i.e., $F(w) \cdot F\mathscr{L} \simeq \mathscr{L}$. It follows that for any $w_i \in Z_i$, $(0 \leqslant i \leqslant r-1)$, we have $w_i F^{i+1}\mathscr{L} \simeq F^i\mathscr{L}$. This implies that

$$\mathscr{L} \simeq w_0 F\mathscr{L} \simeq w_0 w_1 F^2 \mathscr{L} \simeq \cdots \simeq w_0 w_1 \cdots w_{r-1} F^r \mathscr{L}.$$

Since $F^r\mathscr{L} \simeq \mathscr{L}$, (1.4.5) follows.

1.5. Let $\mathbf{w} \in \mathbf{Z}_{\mathscr{L}}$. In view of (1.4.5), one can apply the argument in 1.3 for $\mathbf{w}$ and one obtains the complexes $K_{\dot{\mathbf{w}}}^{\mathscr{L}}$, $\bar{K}_{\dot{\mathbf{w}}}^{\mathscr{L}}$, etc. We choose $\dot{\mathbf{w}} = (\dot{w}_0, \dot{w}_1, \dots, \dot{w}_{r-1})$ so that $\dot{w}_i \in N_G(T)^{F^d}$. Then the complexes $J_{\dot{\mathbf{w}}}^{\mathscr{L}}$, $\hat{J}_{\dot{\mathbf{w}}}^{\mathscr{L}}$, etc. have natural $\mathbf{F}_{q^r}$-structures, which we may regard as the mixed complexes on $\mathbf{F}_{q^r}$-schemes $(\bar{Y}_{\mathbf{w}})_0$, $(\bar{X}_{\dot{\mathbf{w}}})_0$, etc. Following [L3, III, 12], we shall determine the structure of $J_{\dot{\mathbf{w}}}^{\mathscr{L}}$ restricted to various $Y_{\mathbf{y}}$. Note that if we write

$\mathbf{W} = W \times W \times \cdots \times W$, ($r$-factors), $\bar{Y}_{\mathbf{w}}$, (resp. $\bar{X}_{\mathbf{w}}$) is a disjoint union of $Y_{\mathbf{y}}$ (resp. $X_{\mathbf{y}}$) for various $\mathbf{y} \in \mathbf{W}$ such that $\mathbf{y} \leqslant \mathbf{w}$ for the partial order of $\mathbf{W}$. In order to state the next result, we prepare some notation. The length function l: $\mathbf{W} \to \mathbf{N}$ is defined in a usual way. We define a length function $\tilde{l}$: $\mathbf{Z}_{\mathscr{L}} \to \mathbf{N}$ as follows; $Z_{\mathscr{L}}$ has a unique element z_1 of minimal length in W, and is written as $Z_{\mathscr{L}} = z_1 W_{\mathscr{L}}$. Let $\tilde{l}$: $W_{\mathscr{L}} \to \mathbf{N}$ be the length function of the Coxeter group $W_{\mathscr{L}}$, and extend it to $Z_{\mathscr{L}}$ by $\tilde{l}(z_1 w) = \tilde{l}(w)$. This allows us to define $\tilde{l}$: $Z_i \to \mathbf{N}$ for each i, and we define $\tilde{l}$: $\mathbf{Z}_{\mathscr{L}} \to \mathbf{N}$ by $\tilde{l}(\mathbf{w}) = \sum_{i=0}^{r-1} \tilde{l}(w_i)$. Let us define a subgroup $\mathbf{W}_{\mathscr{L}}$ of $\mathbf{W}$ by

$$\mathbf{W}_{\mathscr{L}} = W_{F\mathscr{L}} \times W_{F^2\mathscr{L}} \times \cdots \times W_{F^{r-1}\mathscr{L}} \times W_{F^r\mathscr{L}}.$$

Then $\mathbf{Z}_{\mathscr{L}}$ is written as $\mathbf{Z}_{\mathscr{L}} = \mathbf{w}_1 \mathbf{W}_{\mathscr{L}}$ for $\mathbf{w}_1 = (F(z_1), F^2(z_1), ..., F^r(z_1))$. We choose a coherent lifting $\dot{Z}_i$ (over $\mathbf{F}_{q^d}$) for each Z_i (cf. [L3, III, 12.3]) and put $\dot{\mathbf{Z}}_{\mathscr{L}} = \dot{Z}_0 \times \cdots \times \dot{Z}_{r-1}$. The following result is an analogy of Theorem 12.4 in [L3, III].

Proposition 1.6. *Assume that we are in the setting of 1.5. In particular, $\mathbf{w} \in \mathbf{Z}_{\mathscr{L}}$, and $\dot{\mathbf{Z}}_{\mathscr{L}}$ is a coherent lifting of $\mathbf{Z}_{\mathscr{L}}$. Then for any $\mathbf{y} \in \mathbf{W}$ such that $\mathbf{y} \leqslant \mathbf{w}$, the restriction $\mathscr{H}^i(J_{\dot{\mathbf{w}}}^{\mathscr{L}}) \mid Y_{\mathbf{y}}$ is a local system with finite monodromy. It is zero unless $\mathbf{y} \in \mathbf{Z}_{\mathscr{L}}$ and i is even. If these conditions are satisfied and if $\dot{\mathbf{y}}, \dot{\mathbf{w}} \in \dot{\mathbf{Z}}_{\mathscr{L}}$, it admits a filtration defined over $\mathbf{F}_{q^d}$ by local systems, with all subquotients isomorphic (over $\mathbf{F}_{q^d}$) to $\tilde{\mathscr{P}}_{\dot{\mathbf{y}}} \otimes \bar{\mathbf{Q}}_l(-i/2)$ and with a number of steps equal to $n_{\mathbf{y}, \mathbf{w}, i}$. Here $n_{\mathbf{y}, \mathbf{w}, i}$ is the coefficient of $u^{i/2}$ in*

$$u^{(l(\mathbf{w}) - l(\mathbf{y}) - \tilde{l}(\mathbf{w}) + \tilde{l}(\mathbf{y}))/2} P_{\mathbf{w}_1^{-1}\mathbf{y}, \mathbf{w}_1^{-1}\mathbf{w}}(u),$$

where $P_{\alpha, \beta}(u)$ are Kazhdan–Lusztig polynomials for the Coxeter group $\mathbf{W}_{\mathscr{L}}$.

Proof. We shall prove the proposition along the similar line as in [L3, III]. In the following proof, we use the symbols such as $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$, etc. to indicate the direct product of r-factors of G, B, T, etc., respectively. For $\mathbf{w} \in \mathbf{W}_{\mathscr{L}}$, put $N_{\mathbf{w}} = \mathbf{B}\mathbf{w}\mathbf{B} \times k^*$. We define $\boldsymbol{\lambda} \in \mathrm{Hom}(\mathbf{B}, k^*)$ by $\boldsymbol{\lambda} = F(\lambda) \otimes F^2(\lambda) \otimes \cdots \otimes F^r(\lambda)$. Let $\mathscr{L} \in \mathscr{S}(\mathbf{T})$ be the tame local system corresponding to $(\boldsymbol{\lambda}, n)$ under the isomorphism (1.4.1) applied to $\mathbf{T}$. Then $\mathbf{W}_{\mathscr{L}}$ is the subgroup of $\mathbf{W}$ associated to $\mathscr{L}$ defined in a similar way as $W_{\mathscr{L}}$. We define an action of $\mathbf{B}$ on $N_{\mathbf{w}}$ by $\mathbf{b}$: $(\mathbf{g}, z) \mapsto (\mathbf{g}\mathbf{b}^{-1}, \boldsymbol{\lambda}(\mathbf{b})\, z)$. Put $P_{\mathbf{w}} = N_{\mathbf{w}}/\mathbf{B}$, and let p: $N_{\mathbf{w}} \to P_{\mathbf{w}}$ be the canonical map. Consider the principal μ_n-covering

$$\mathbf{U}/\mathbf{U} \cap \dot{\mathbf{w}}\mathbf{U}\dot{\mathbf{w}}^{-1} \times k^* \to P_{\mathbf{w}}, \qquad (\mathbf{u}, z) \mapsto \mathbf{B}\text{-orbit of } (\mathbf{u}\dot{\mathbf{w}}, z^n).$$

We denote by $\mathscr{E}_{\dot{\mathbf{w}}}$ the irreducible local system on $P_{\mathbf{w}}$ (defined over $\mathbf{F}_{q^d}$) attached to the above μ_n-covering and to ψ: $\mu_n \to \bar{\mathbf{Q}}_l^*$. Let $\bar{N}_{\mathbf{w}} = \overline{\mathbf{B}\mathbf{w}\mathbf{B}} \times k^*$, $\bar{P}_{\mathbf{w}} = \bar{N}_{\mathbf{w}}/\mathbf{B}$, under the $\mathbf{B}$-action on $\bar{N}_{\mathbf{w}}$ similar to $N_{\mathbf{w}}$. Then $N_{\mathbf{w}}$ is open dense in $\bar{N}_{\mathbf{w}}$, and so $P_{\mathbf{w}}$ is open dense in $\bar{P}_{\mathbf{w}}$. We have $N_{\mathbf{y}} \subset \bar{N}_{\mathbf{w}}$, $P_{\mathbf{y}} \subset \bar{P}_{\mathbf{w}}$ for $\mathbf{y} \in \mathbf{W}$ such that $\mathbf{y} \leqslant \mathbf{w}$. Note that $N_{\mathbf{y}}$, $P_{\mathbf{y}}$ can be defined for any $\mathbf{y} \in \mathbf{W}$. We consider the intersection cohomology $\mathrm{IC}(\bar{P}_{\mathbf{w}}, \mathscr{E}_{\dot{\mathbf{w}}})$ on $P_{\mathbf{w}}$. It follows from [L3, III, (12.4.2)] applied to $\mathbf{G}$ and $\mathbf{Z}_{\mathscr{L}}$, we see that $\mathrm{IC}(\bar{P}_{\mathbf{w}}, \mathscr{E}_{\dot{\mathbf{w}}})$ satisfies the similar property as the statement for $J_{\mathbf{w}}^{\mathscr{L}}$ in the proposition, i.e., the following holds; the restriction of $\mathscr{H}^i(\mathrm{IC}(\bar{P}_{\mathbf{w}}, \mathscr{E}_{\dot{\mathbf{w}}}))$ to $P_{\mathbf{y}}$ is a local system (defined over $\mathbf{F}_{q^d}$) with finite monodromy. It is zero unless $\mathbf{y} \in \mathbf{Z}_{\mathscr{L}}$ and i is even. If these conditions are satisfied, it admits a filtration by local systems (defined over $\mathbf{F}_{q^d}$) with all subquotients isomorphic (over $\mathbf{F}_{q^d}$) to $\mathscr{E}_{\dot{\mathbf{y}}} \otimes \bar{\mathbf{Q}}_l(-i/2)$ (with $\dot{\mathbf{y}} \in \dot{\mathbf{Z}}_{\mathscr{L}}$) and with a number of steps equal to $n_{\mathbf{w}_1^{-1}\mathbf{y}, \mathbf{w}_1^{-1}\mathbf{w}, i}$. As is discussed in [*loc. cit.*], we shall deduce the proposition from this property of $\mathrm{IC}(\bar{P}_{\mathbf{w}}, \mathscr{E}_{\dot{\mathbf{w}}})$.

Consider the diagram,

$$Y_{\mathbf{w}} \xleftarrow{\beta_{\mathbf{w}}} X_{\mathbf{w}} \xleftarrow{\mathrm{pr}_1} X_{\mathbf{w}} \times k^* \xrightarrow{\theta} N_{\mathbf{w}} \xrightarrow{p} P_{\mathbf{w}},$$

where

$$\theta((g, h_0, h_1, ..., h_r), z) = (h_0^{-1}h_1, h_1^{-1}h_2, ..., h_{r-1}^{-1}h_r, z^n).$$

We also consider the similar diagram replacing the varieties $X_{\mathbf{w}}$, $N_{\mathbf{w}}$, etc. by $\bar{X}_{\mathbf{w}}$, $\bar{N}_{\mathbf{w}}$, etc. Then $p \circ \theta$: $\bar{X}_{\mathbf{w}} \times k^* \to \bar{P}_{\mathbf{w}}$ is a locally trivial fibration with smooth fibres, and $\bar{\beta}_{\mathbf{w}} \circ \mathrm{pr}_1$: $\bar{X}_{\mathbf{w}} \times k^* \to \bar{Y}_{\mathbf{w}}$ is a locally trivial fibration with smooth and connected fibres. Thus by the similar argument as in the proof of Theorem 12.4 in [*loc. cit.*], the proof of the proposition is reduced to show the following.

(1.6.1) $(\beta_{\mathbf{w}} \circ \mathrm{pr}_1)^* \tilde{\mathscr{L}}_{\dot{\mathbf{w}}}$ is isomorphic to $(p \circ \theta)^* \mathscr{E}_{\dot{\mathbf{w}}}$, as local systems (defined over $\mathbf{F}_{q^d}$) on $X_{\mathbf{w}} \times k^*$.

We shall show (1.6.1). We note that $\beta_{\mathbf{w}}^* \tilde{\mathscr{L}}_{\mathbf{w}} = \hat{\mathscr{L}}_{\dot{\mathbf{w}}}$, and that $\hat{\mathscr{L}}_{\dot{\mathbf{w}}}$ is the local system associated to the following principal μ_n-covering of $X_{\mathbf{w}}$ and to ψ.

$$\{(x, \xi) \in X_{\mathbf{w}} \times k^* \mid \lambda(\alpha_{\dot{\mathbf{w}}}(x)) = \xi^n\} \to X_{\mathbf{w}}, \qquad (x, \xi) \mapsto x.$$

Thus, $(\beta_{\mathbf{w}} \circ \mathrm{pr}_1)^* \tilde{\mathscr{L}}_{\dot{\mathbf{w}}} = \mathrm{pr}_1^* \hat{\mathscr{L}}_{\dot{\mathbf{w}}}$ is the local system on $X_{\mathbf{w}} \times k^*$ associated to the μ_n-covering γ: $\tilde{X}_{\mathbf{w}} \to X_{\mathbf{w}} \times k^*$, and to ψ, where

$$\begin{aligned} \tilde{X}_{\mathbf{w}} &= \{(x, \xi, z) \in X_{\mathbf{w}} \times k^* \times k^* \mid \lambda(\alpha_{\dot{\mathbf{w}}}(x)) = \xi^n\}, \\ \gamma(x, \xi, z) &= (x, z). \end{aligned} \tag{1.6.2}$$

On the other hand, by (12.4.5) in [*loc. cit.*], $p^*\mathscr{E}_{\dot{\mathbf{w}}}$ is the local system on $N_{\mathbf{w}}$ associated to the following principal μ_n-covering of $N_{\mathbf{w}}$ and to ψ.

$$\varphi: N_{\mathbf{w}} \to N_{\mathbf{w}}, \qquad (\mathbf{g}, z) \mapsto (\mathbf{g}, \boldsymbol{\lambda}(\mathrm{pr}_{\dot{\mathbf{w}}}(\mathbf{g}))^{-1} z^n),$$

where $\mathrm{pr}_{\mathbf{w}}$: $\mathbf{BwB} \to \mathbf{T}$ is the projection to the $\mathbf{T}$-factor, $\mathbf{u\dot{w}tv} \mapsto \mathbf{t}$. Next we shall describe the local system $\theta^* p^* \mathscr{E}_{\dot{\mathbf{w}}}$ on $X_{\mathbf{w}} \times k^*$. Consider the diagram,

$$\begin{tikzcd} \tilde{X}'_{\mathbf{w}} \arrow[r, "\delta"] \arrow[d, "\varepsilon"'] & N_{\mathbf{w}} \arrow[d, "\varphi"] \\ X_{\mathbf{w}} \times k^* \arrow[r, "\theta"] & N_{\mathbf{w}}, \end{tikzcd} \tag{1.6.3}$$

where

$$\tilde{X}'_{\mathbf{w}} = \{ ((g, h_0, h_1, ..., h_r), \xi, z) \in X_{\mathbf{w}} \times k^* \times k^* \\ \mid \boldsymbol{\lambda}(\mathrm{pr}_{\dot{w}_0}(h_0^{-1} h_1), \mathrm{pr}_{\dot{w}_1}(h_1^{-1} h_2), ..., \mathrm{pr}_{\dot{w}_{r-1}}(h_{r-1}^{-1} h_r)) = \xi^n \},$$

and

$$\delta((g, h_0, h_1, ..., h_r), \xi, z) = (h_0^{-1} h_1, h_1^{-1} h_2, ..., h_{r-1}^{-1} h_r, z\xi),$$
$$\varepsilon((g, h_0, h_1, ..., h_r), \xi, z) = ((g, h_0, h_1, ..., h_r), z).$$

Here $\mathrm{pr}_{\dot{w}}$: $BwB \to T$ is the projection on the T-factor, $u\dot{w}tv \mapsto t$ for $w \in W$. Now it is easy to see that this diagram is cartesian. Hence $\theta^* p^* \mathscr{E}_{\dot{\mathbf{w}}}$ is the local system associated to the μ_n-covering ε: $\tilde{X}'_{\mathbf{w}} \to X_{\mathbf{w}} \times k^*$ and to ψ. It follows from (1.6.2) and (1.6.3), we see that (1.6.1) is deduced from the following statement.

(1.6.4) The μ_n-covering γ: $\tilde{X}_{\mathbf{w}} \to X_{\mathbf{w}} \times k^*$ is isomorphic (over $\mathbf{F}_{q^r}$) to the μ_n-covering ε: $\tilde{X}'_{\mathbf{w}} \to X_{\mathbf{w}} \times k^*$.

We show (1.6.4) by constructing an isomorphism f: $\tilde{X}'_{\mathbf{w}} \to \tilde{X}_{\mathbf{w}}$ such that $\varepsilon = \gamma \circ f$. First we note that

(1.6.5) There exist $\lambda_i \in \mathrm{Hom}(T, k^*)$ for $i = 0, 1, ..., r-1$ such that

$$w_{i+1} w_{i+2} \cdots w_{r-1} \cdot \lambda = F^{i+1}(\lambda)\, \lambda_i^n.$$

(For $i = r-1$, it simply means that $\lambda = F^r(\lambda)\, \lambda_{r-1}^n$.)

In fact, since $w_i \in Z_i$, we have $w_i F^{i+1} \mathscr{L} \simeq F^i \mathscr{L}$ (see the proof of (1.4.5)). Using $F^r \mathscr{L} \simeq \mathscr{L}$, we get

$$\begin{aligned} w_{i+1} w_{i+2} \cdots w_{r-1} \mathscr{L} &\simeq w_{i+1} \cdots w_{r-2} F^{r-1} \mathscr{L} \\ &\simeq w_{i+1} \cdots w_{r-3} F^{r-2} \mathscr{L} \\ &\cdots \\ &\simeq F^{i+1} \mathscr{L}. \end{aligned}$$

This implies (1.6.5) in view of (1.4.1).

We define $f\colon \tilde{X}'_{\mathbf{w}} \to \tilde{X}_{\mathbf{w}}$ by

$$((g, h_0, ..., h_r), \xi, z) \mapsto \left((g, h_0, ..., h_r), \xi \cdot \prod_{i=0}^{r-1} \lambda_i(t_i), z\right),$$

where $t_i = \mathrm{pr}_{\dot{w}_i}(h_i^{-1} h_{i+1})$, $(0 \leqslant i \leqslant r-1)$. We note that the right hand side of (1.6.5) is actually in $\tilde{X}_{\mathbf{w}}$. In fact, since $\lambda(t_0, ..., t_{r-1}) = \xi^n$, we have $\prod_{i=0}^{r-1} F^{i+1}(\lambda)(t_i) = \xi^n$. On the other hand,

$$\begin{aligned}\alpha_{\dot{\mathbf{w}}}((g, h_0, ..., h_r)) &= (\dot{w}_0 \cdots \dot{w}_{r-1})^{-1} (\dot{w}_0 t_0 \cdot \dot{w}_1 t_1 \cdots \dot{w}_{r-1} t_{r-1}) \\ &= \prod_{i=0}^{r-1} (\dot{w}_{i+1} \cdots \dot{w}_{r-1})^{-1} t_i (\dot{w}_{i+1} \cdots \dot{w}_{r-1}).\end{aligned}$$

Hence, we have

$$\begin{aligned}\lambda(\alpha_{\dot{\mathbf{w}}}(g, h_0, ..., h_r)) &= \prod_{i=0}^{r-1} (w_{i+1} \cdots w_{r-1} \cdot \lambda)(t_i) \\ &= \prod_{i=0}^{r-1} F^{i+1}(\lambda)(t_i) \cdot \prod_{i=0}^{r-1} \lambda_i^n(t_i) \qquad \text{by (1.6.5)} \\ &= \left(\xi \cdot \prod_{i=0}^{r-1} \lambda_i(t_i)\right)^n.\end{aligned}$$

Thus f is well-defined. Clearly f is an isomorphism since $\prod_{i=0}^{r-1} \lambda_i(t_i)$ is determined by $(g, h_0, ..., h_r)$. Also f satisfies the relation $\gamma \circ f = \varepsilon$ and f commutes with the action of μ_n. Hence (1.6.4) holds, and so (1.6.1) follows. The proposition is now proved.

As a corollary, we have

COROLLARY 1.7 (cf. [L3, III, Cor. 12.5]). *Let $\mathbf{w} \in \mathbf{Z}_{\mathscr{L}}$ and assume that $\mathbf{y} \leqslant \mathbf{w}$. Then the following identity holds in the Grothendieck group of mixed perverse sheaves on G_0.*

$$\begin{aligned}&{}^pH^j((\pi_{\mathbf{y}})_!(\mathscr{H}^i(J_{\dot{\mathbf{w}}}^{\mathscr{L}}) \mid Y_{\mathbf{y}})) \\ &= \begin{cases} n_{\mathbf{y},\mathbf{w},i}\, {}^pH^j((\pi_{\mathbf{y}})_!\, \tilde{\mathscr{L}}_{\dot{\mathbf{y}}})(-i/2) & \text{if } \dot{\mathbf{y}} \in \dot{\mathbf{Z}}_{\mathscr{L}} \text{ and } i \text{ is even} \\ 0 & \text{if } \mathbf{y} \notin \mathbf{Z}_{\mathscr{L}} \text{ or } i \text{ is odd.}\end{cases}\end{aligned}$$

1.8. Let $\mathbf{w} \in \mathbf{Z}_{\mathscr{L}}$ be as before. For $\mathbf{x} \in \mathbf{W}_{\mathscr{L}}$, put $Y = \bigcup_{\mathbf{z} \leqslant \mathbf{x}} Y_{\mathbf{wz}}$. We assume that Y is contained in $\bar{Y}_{\mathbf{wx}}$, and we denote the inclusion by ϕ: $Y \subsetneq \bar{Y}_{\mathbf{wx}}$. Recall that $\hat{G}_{\mathscr{L}}$ denotes the set of character sheaves on G associated to $\mathscr{L} \in \mathscr{S}(T)$. The following proposition can be proved in a similar way as Proposition 12.7 in [L3, III] by using Corollary 1.7 instead of Corollary 12.5 in [*loc. cit.*], if we notice that any irreducible perverse sheaf is a constituent of ${}^pH^i(K_{\mathbf{y}}^{\mathscr{L}})$ for $\mathbf{y} \in \mathbf{Z}_{\mathscr{L}}$ belongs to $\hat{G}_{\mathscr{L}}$, (cf. [L3, I, Prop. 2.9]).

PROPOSITION 1.9. *Under the assumption of* 1.8, *let* $K = (\pi_{\mathbf{wx}})_! \phi_! \phi^* J_{\mathbf{wx}}^{\mathscr{L}}$. *Then any irreducible perverse sheaf on G which is a constituent of* ${}^pH^i(K)$ *belongs to* $\hat{G}_{\mathscr{L}}$.

1.10. We now return to the setup in 1.1, and consider the twisted induction $\mathbf{w}\text{-ind}_P^G$ for $\mathbf{w} = (w_0, w_1, ..., w_{r-1})$ satisfying (1.1.1). For a tame local system $\mathscr{L} \in \mathscr{S}(T)$, we consider the set of character sheaves $\hat{G}_{\mathscr{L}}$ and $\hat{L}_{\mathscr{L}}$ associated to $\mathscr{L}$. The following proposition is analogous to Proposition 4.2 in [S3].

PROPOSITION 1.11. *Let* $\mathbf{w} = (w_0, w_1, ..., w_{r-1})$ *be as in* 1.10. *Let r be as in* (1.4.4) *and assume that* $\mathbf{w} \in \mathbf{Z}_{\mathscr{L}}$. *Then for each* $A \in \hat{L}_{\mathscr{L}}$, *all the irreducible constituents in* ${}^pH^i(\mathbf{w}\text{-ind}_P^G A)$ *are contained in* $\hat{G}_{\mathscr{L}}$.

Proof. We prove the proposition in a similar way as [S3, Prop. 4.2]. We fix $x \in (W_L)_{\mathscr{L}}$ and consider the diagram,

$$\begin{array}{ccccccc}
\bar{Y}_{x,L} & \xleftarrow{\mathrm{pr}_2} & \hat{V}' \times_L \bar{Y}_{x,L} & \xrightarrow{\hat{\pi}} & Y & & \\
\Big\downarrow{\scriptstyle \pi_x} & & \Big\downarrow{\scriptstyle \mathrm{pr}_1} & & \Big\downarrow{\scriptstyle \zeta} & & \\
L & \xleftarrow{\rho'} & \hat{V}' & \xrightarrow{\pi''} & V & \xrightarrow{\pi} & G,
\end{array} \tag{1.11.1}$$

where the lower horizontal arrows are as in 1.1 and 1.2, and $Y = \bigcup_{z \leqslant x} Y_{\mathbf{wz}}$, with $\mathbf{z} = (1, ..., 1, z) \in \mathbf{W}_L$ (see 1.3),

$$\bar{Y}_{x,L} = \{(l, yB_L) \in L \times L/B_L \mid y^{-1}ly \in \overline{B_L x B_L}\},$$

$$\zeta(g, h_0 B, ..., h_r B) = (g, h_0 P, ..., h_r P), \qquad \bar{\pi}_x(l, yB_L) = l.$$

$\hat{\pi}$ is defined as

$$\begin{aligned}
&\hat{\pi}(g, h_0 U_P, ..., h_{r-1} U_P, l, yB_L) \\
&\quad = (g, h_0 y_0 B, h_1 y_1 B, ..., h_{r-1} y_{r-1} B, gh_0 y_0 B),
\end{aligned}$$

where $y_0 = y$, $y_i = (\dot{w}_0 \dot{w}_1 \cdots \dot{w}_{i-1})^{-1} y (\dot{w}_0 \dot{w}_1 \cdots \dot{w}_{i-1}) \in W_L$, $(1 \leqslant i \leqslant r-1)$. We note that $\hat{\pi}$ is well defined. In fact, we have $h_i^{-1} h_{i+1} \in U_P \dot{w}_i U_P$, $(0 \leqslant i \leqslant r-2)$, and $h_{r-1}^{-1} g h_0 \in U_P \dot{w}_{r-1} l' U_P$ for $l' \in L$ such that $l = \dot{w}_0 \dot{w}_1 \cdots \dot{w}_{r-1} l'$. Hence, for $i = 0, 1, ..., r-2$, we have

$$\begin{aligned}
(h_i y_i)^{-1} (h_{i+1} y_{i+1}) &\in y_i^{-1} U_P \dot{w}_i U_P y_{i+1} \\
&= U_P y_i^{-1} \dot{w}_i y_{i+1} U_P \\
&= U_P \dot{w}_i U_P \\
&\subset B \dot{w}_i B,
\end{aligned}$$

and

$$\begin{aligned}(h_{r-1}y_{r-1})^{-1}(gh_0y_0) &\in y_{r-1}^{-1}U_P\dot{w}_{r-1}l'U_Py_0\\ &= U_P(\dot{w}_0\cdots\dot{w}_{r-2})^{-1}y^{-1}(\dot{w}_0\cdots\dot{w}_{r-1})l'yU_P\\ &\in Bw_{r-1}y^{-1}lyB \qquad (\text{since } \dot{w}_0\cdots\dot{w}_{r-1}\in T)\\ &\subset \bigcup_{z\leqslant x} Bw_{r-1}zB.\end{aligned}$$

Thus $\hat{\pi}$ is well defined. We note also that the square in the right side of (1.11.1) is cartesian. In fact, the map $\hat{V}'\times_V Y\to\hat{V}'\times_L\bar{Y}_{x,L}$ which is inverse to the natural map $\hat{V}'\times_L\bar{Y}_{x,L}\to\hat{V}'\times_V Y$ is constructed as follows. Take

$$\alpha=(g,h_0U_P,...,h_{r-1}U_P)\in\hat{V}',$$

$$\beta=(g',h'_0B,...,h'_{r-1}B,h'_rB)\in Y$$

such that $\pi'(\alpha)=\zeta(\beta)$. So, we have $g=g'$ and $h_iP=h'_iP$ for $i=0,...,r-1$. One can find l_i (unique modulo B_L) such that $h'_iB=h_il_iB$ for $i=0,...,r-1$. Put

$$l'_i=(\dot{w}_0\dot{w}_1\cdots\dot{w}_{i-1})\,l_i(\dot{w}_0\dot{w}_1\cdots\dot{w}_{i-1})^{-1},$$

for $i=1,...,r-1$, and put $l'_0=l_0$. These l'_i are also determined uniquely modulo B_L since w_i normalizes B_L. Then we have, for $i=0,...,r-2$,

$$\begin{aligned}(h_il_i)^{-1}(h_{i+1}l_{i+1}) &\in Bw_iB\cap U_Pl_i^{-1}\dot{w}_il_{i+1}U_P\\ &= Bw_iB\cap U_P\dot{w}_i(\dot{w}_0\cdots\dot{w}_i)^{-1}l_i'^{-1}l'_{i+1}(\dot{w}_0\cdots\dot{w}_i)\,U_P.\end{aligned}$$

Hence $l_i'^{-1}l'_{i+1}\in B_L$ and we see that $l_0B_L=l'_1B_L=\cdots=l'_{r-1}B_L$. Moreover since $\alpha\in\hat{V}'$, we have $h_{r-1}^{-1}gh_0\in U_P\dot{w}_{r-1}l'U_P$ for $l'\in L$. It follows, if we write $l=\dot{w}_0\dot{w}_1\cdots\dot{w}_{r-1}l'\in L$, that

$$\begin{aligned}(h_{r-1}l_{r-1})^{-1}(gh_0l_0) &\in Bw_{r-1}zB\cap U_Pl_{r-1}^{-1}\dot{w}_{r-1}l'l_0U_P\\ &= Bw_{r-1}zB\cap U_P(\dot{w}_0\cdots\dot{w}_{r-2})^{-1}(l'_{r-1})^{-1}ll_0U_P,\end{aligned}$$

and so, $(l'_{r-1})^{-1}ll_0\in B_LzB_L$. Therefore $\gamma=(l,l_0B_L)\in\bar{Y}_{x,L}$, and it is easily verified that the correspondence $(\alpha,\beta)\mapsto(\alpha,\gamma)$ gives the required map $\hat{V}'\times_V Y\to\hat{V}'\times_L\bar{Y}_{x,L}$.

Now put $\mathbf{x}=(1,...,1,x)\in\mathbf{W}_{\mathscr{L}}$ and consider the variety $\bar{Y}_{\mathbf{wx}}$, the complex $J^{\mathscr{L}}_{\dot{\mathbf{w}}\dot{\mathbf{x}}}=\mathrm{IC}(\bar{Y}_{\mathbf{wx}},\bar{\mathscr{L}}_{\dot{\mathbf{w}}\dot{\mathbf{x}}})$ on $\bar{Y}_{\mathbf{wx}}$ as in 1.5. We have the diagram,

$$\begin{tikzcd} \bar{Y}_{\mathbf{wx}} \arrow[r, "\bar{\zeta}"] & \bar{V} & \\ Y \arrow[u, hook, "j_Y"] \arrow[r, "\zeta"] & V \arrow[u, hook, "j"'] \arrow[r, "\pi"] & G, \end{tikzcd} \qquad (1.11.2)$$

where

$$\bar{V} = \{(g, h_0P, h_1P, \dots, h_rP) \in G \times G/P \times G/P \times \cdots \times G/P \mid h_i^{-1}h_{i+1} \in P\dot{w}_iP \ (0 \leqslant i \leqslant r-1),\ h_r^{-1}gh_0 \in P\}$$

and $\bar{\zeta}(g, h_0B, \dots, h_rB) = (g, h_0P, \dots, h_rP)$. In a similar argument as in the proof of (4.2.2) in [S3], we see that $\bar{\zeta}^{-1}(V) = Y$, i.e., the diagram in (1.11.2) is cartesian. Put $M_{\mathbf{w}\mathbf{x}} = \bar{\zeta}_! J^{\mathscr{L}}_{\mathbf{w}\mathbf{x}} \in \mathscr{D}\bar{V}$. We will work in the derived category of mixed complexes in G_0. Then $J^{\mathscr{L}}_{\mathbf{w}\mathbf{x}}$ is a pure complex of weight 0. Since $\bar{\zeta}$ is proper, $M_{\mathbf{w}\mathbf{x}}$ is pure. Since $j: V \subsetneq \bar{V}$ is an open embedding, $j^*M_{\mathbf{x}\mathbf{x}}$ is pure. Thus $j^*M_{\mathbf{x}\mathbf{x}}$ is semisimple and we have the decomposition

$$j^*M_{\mathbf{w}\mathbf{x}} = \bigoplus_i {}^pH^i(j^*M_{\mathbf{w}\mathbf{x}})[-i]. \tag{1.11.3}$$

Put $J^{\vee}_{\mathbf{w}\mathbf{x}} = J^{\mathscr{L}}_{\mathbf{w}\mathbf{x}} \mid Y$. Then we have $J^{\vee}_{\mathbf{w}\mathbf{x}} = \mathrm{IC}(Y, \tilde{\mathscr{T}}_{\mathbf{w}\mathbf{x}})$. It follows from the cartesian diagram in (1.11.2) that

$$j^*M_{\mathbf{w}\mathbf{x}} = \zeta_! J^{\vee}_{\mathbf{w}\mathbf{x}}. \tag{1.11.4}$$

We define a complex $K^{\vee}_{\mathbf{w}\mathbf{x}} = \pi_!\zeta_! J^{\vee}_{\mathbf{w}\mathbf{x}} \in \mathscr{D}G$. Applying ${}^pH^j\pi_!$ on both side of (1.11.3), and using (1.11.4), we have

$${}^pH^j(K^{\vee}_{\mathbf{w}\mathbf{x}}) = \bigoplus_i {}^pH^{j-i}\pi_!({}^pH^i(\zeta_! J^{\vee}_{\mathbf{w}\mathbf{x}})). \tag{1.11.5}$$

Let $J^{\mathscr{L}}_{x,L} \in \mathscr{D}\bar{Y}_{x,L}$, $\bar{K}^{\mathscr{L}}_{x,L} = (\bar{\pi}_x)_! J^{\mathscr{L}}_{x,L} \in \mathscr{D}L$ be the complexes defined in [S3, 1.3] with $\sigma = 1$, replacing G by L, (i.e., the complexes defined in 1.3 in the special case where $\mathbf{w}$ is reduced to one element $x \in W_{\mathscr{L}}$). Then by the similar argument as in the proof of Proposition 4.2 in [*loc. cit.*], we see easily that

$$\mathrm{pr}_2^* J^{\mathscr{L}}_{x,L} = \mathrm{IC}(\hat{V}' \times_L \bar{Y}_{x,L}, (\hat{\pi}')^* \tilde{\mathscr{T}}_{\mathbf{w}\mathbf{x}}) = \hat{\pi}^* J^{\vee}_{\mathbf{w}\mathbf{x}},$$

where $\hat{\pi}': \hat{V}' \times_L Y_{x,L} \to Y_{\mathbf{w}\mathbf{x}}$ is the restriction of $\hat{\pi}$ to $\hat{V}' \times_L Y_{x,L}$. Using the cartesian squares in (1.11.1), we have

$$(p')^* \bar{K}^{\mathscr{L}}_{x,L} = (\mathrm{pr}_1)_! \mathrm{IC}(\hat{V}' \times_L \bar{Y}_{x,L}, (\hat{\pi}')^* \tilde{\mathscr{T}}_{\mathbf{w}\mathbf{x}}) = (\pi'')^* \zeta_! J^{\vee}_{\mathbf{w}\mathbf{x}}.$$

Put

$$d_{\mathbf{w}} = \dim G + \sum_{i=0}^{r-1} \dim U_{w_i} - r \cdot \dim L, \tag{1.11.6}$$

where $U_{w_i} = U \cap w_i^{-1}U^- w_i \subset U_P$ for the opposite unipotent subgroup U^- of U with respect to B, T. Then we have

$$\tilde{p}'\bar{K}^{\mathscr{L}}_{x,L} = \tilde{\pi}''\zeta_! J^{\vee}_{\mathbf{w}\mathbf{x}}[d_{\mathbf{w}}].$$

In fact this follows from the fact that the dimension of the fibre of p' is $\dim G + \sum \dim U^-_{w_i}$ and the dimension of the fibre of π'' is $r \cdot \dim L$. Now using the fact that $\tilde{p}'$ and $\tilde{\pi}''$ are t-exact, we have

$$\tilde{p}'({}^pH^i(\bar{K}^{\mathscr{L}'}_{\check{x},L})) = \tilde{\pi}''({}^pH^{i+d_{\mathbf{w}}}(\zeta_! J^{\vee}_{\dot{\mathbf{w}}\check{\mathbf{x}}})).$$

Then it follows from the definition of the twisted induction $\mathbf{w}\text{-ind}^G_P$ that we have

$$\mathbf{w}\text{-ind}^G_P({}^pH^i\bar{K}^{\mathscr{L}'}_{\check{x},L}) = \pi_!({}^pH^{i+d_{\mathbf{w}}}\zeta_! J^{\vee}_{\dot{\mathbf{w}}\check{\mathbf{x}}}).$$

Hence we have

$$\bigoplus_i {}^pH^{j-i}(\mathbf{w}\text{-ind}^G_P({}^pH^i\bar{K}^{\mathscr{L}'}_{\check{x},L})) = \bigoplus_i {}^pH^{j-i}\pi_!({}^pH^{i+d_{\mathbf{w}}}\zeta_! J^{\vee}_{\dot{\mathbf{w}}\check{\mathbf{x}}}) = {}^pH^{j+d_{\mathbf{w}}}K^{\vee}_{\dot{\mathbf{w}}\check{\mathbf{x}}}$$

by (1.11.5). Thus, in order to prove the proposition, it is enough to show that each constituent of ${}^pH^jK^{\vee}_{\dot{\mathbf{w}}\check{\mathbf{x}}}$ is contained in $\hat{G}_{\mathscr{L}'}$. But since

$$K^{\vee}_{\dot{\mathbf{w}}\check{\mathbf{x}}} = (\bar{\pi}_{\mathbf{w}\check{\mathbf{x}}})_!(j_Y)_! j_Y^* J^{\mathscr{L}'}_{\dot{\mathbf{w}}\check{\mathbf{x}}},$$

Proposition 1.9 can be applied to get the required result. This proves the proposition.

2. Shintani Descent Identities for Character Sheaves

2.1. Let G, B, T, etc. be as in 1.1. We assume that T, B, L, and P are F-stable. We assume further that there exists an $\mathbf{F}_q$-split structure on G with Frobenius map F_0 commuting with F, and that T, B, L, and P are all F_0-stable. Thus F and F_0 acts naturally on $N_W(W_L)$. Let $w \in N_W(W_L)$ be such that $wB_Lw^{-1} = B_L$, and choose a representative $\dot{w} \in N_G(L)^{F_0}$ of w. We also choose an integer $r \geqslant 1$ so that

$$F^r(\dot{w}) = \dot{w}, \tag{2.1.1}$$

$$wF(w)F^2(w)\cdots F^{r-1}(w) = 1 \text{ in } W. \tag{2.1.2}$$

Let $\mathbf{w} = (w_0, w_1, ..., w_{r-1}) = (F(w), F^2(w), ..., F^r(w))$ be a sequence of elements in W, and put $\dot{\mathbf{w}} = (F(\dot{w}), F^2(\dot{w}), ..., F^r(\dot{w}))$. We shall consider the twisted induction $\mathbf{w}\text{-ind}^G_P$ as defined in Section 1. Let V, $\hat{V}'$ be as in Section 1. In order to construct characteristic functions such as [L4, 5.5] (cf. [S3, Sect. 2]), we shall define morphisms $\bar{F}\colon V \to V$, $\hat{F}\colon \hat{V}' \to \hat{V}'$ as follows;

$$\begin{aligned}&\bar{F}(g, h_0P, h_1P, ..., h_rP)\\ &\qquad = (F(g), F(g^{-1}h_{r-1})P, F(h_0)P, ..., F(h_{r-1})P),\end{aligned}$$

$$\begin{aligned}\hat F(g, h_0 U_P, h_1 U_P, ..., h_{r-1} U_P)\\ = (F(g), F(g^{-1}h_{r-1}\dot w_{r-1} l \dot w_{r-1}^{-1}) U_P, F(h_0) U_P, ..., F(h_{r-2}) U_P),\end{aligned}$$

where $l \in L$ is defined by the condition that $h_{r-1}^{-1} g h_0 \in U_P \dot w_{r-1} l U_P$. It is easy to check that $\bar F$, $\hat F$ are well-defined. We have the following commutative diagram, (cf. [L4, Lemma 5.4]).

$$\begin{tikzcd} L \arrow[d, "F\dot w"'] & \hat V' \arrow[l, "p'"'] \arrow[r, "\pi''"] \arrow[d, "\hat F"] & V \arrow[r, "\pi"] \arrow[d, "\bar F"] & G \arrow[d, "F"] \\ L & \hat V' \arrow[l, "p'"'] \arrow[r, "\pi''"] & V \arrow[r, "\pi"] & G, \end{tikzcd} \tag{2.1.3}$$

where p' and π'' are as in 1.2. In fact the commutativity of the right square and the middle square is obvious from the definition of $\bar F$ and $\hat F$. We verify the commutativity of the left square. Take $\alpha = (g, h_0 U_P, ..., h_{r-1} U_P) \in \hat V'$. Then $p'(\alpha) = \dot w_0 \dot w_1 \cdots \dot w_{r-1} l$ with $l \in L$ such that $h_{r-1}^{-1} g h_0 \in U_P \dot w_{r-1} l U_P$. Since $\dot w_{r-1} = \dot w$, we have

$$F(\dot w(p'(\alpha)) \dot w^{-1}) = \dot w_0 \dot w_1 \cdots \dot w_{r-1} \cdot \dot w_0 F(l) \dot w_0^{-1}.$$

But since

$$F(h_{r-2})^{-1} F(g) F(g^{-1} h_{r-1} \dot w_{r-1} l \dot w_{r-1}^{-1}) \in U_P \dot w_{r-1} \dot w_0 F(l) \dot w_0^{-1} U_P,$$

we have $p'(\hat F(\alpha)) = \dot w_0 \cdots \dot w_{r-1} \cdot \dot w_0 F(l) \dot w_0^{-1}$. This shows the commutativity.

Now, F is written as $F = F_0 \sigma$, where σ is the graph automorphism of G leaving T, B, and L invariant. We shall decompose $\hat F$ and $\bar F$ according to the decomposition $F = F_0\sigma$ as discussed in [S3, Sect. 2]. The varieties V and $\hat V'$ have natural $\mathbf{F}_q$-structures, given by the Frobenius maps,

$$\begin{aligned} (g, h_0 P, ..., h_r P) &\mapsto (F_0(g), F_0(h_0) P, ..., F_0(h_r) P), \qquad V \to V\\ (g, h_0 U_P, ..., h_{r-1} U_P) &\mapsto (F_0(g), F_0(h_0) U_P, ..., F_0(h_{r-1}) U_P), \qquad \hat V' \to \hat V', \end{aligned}$$

both of which we denote by F_0. Let us define a map $\tau\colon V \to V$ by

$$\tau(g, h_0 P, ..., h_r P) = (\sigma(g), \sigma(g^{-1} h_{r-1}) P, \sigma(h_0) P, ..., \sigma(h_{r-1}) P).$$

Since σ permutes all the components in $\mathbf{w}$ in a cyclic way, we see easily that τ is well defined, and commutes with F_0. Moreover τ is an automorphism on V; the inverse of τ is given by

$$(g, h_0 P, ..., h_r P) \mapsto (\sigma^{-1}(g), \sigma^{-1}(h_1) P, ..., \sigma^{-1}(h_r) P, \sigma^{-1}(g h_1) P).$$

We note that $\bar F$ coincides with τF_0. It follows that $\bar F$ is a finite morphism on V.

2.2. We fix a character sheaf $A_0 \in \hat{L}_{\mathscr{L}}$. Let us choose $w \in W$ such that $(F\dot{w})^* \mathscr{L} \simeq \mathscr{L}$. We assume that w satisfies the assumption of 2.1, and choose an integer r large enough so that it satisfies (1.4.4), (2.1.1) and (2.2.2). Let $K_{\mathbf{w}}$ be the twisted induction $\mathbf{w}\text{-ind}_P^G A_0$ for $\mathbf{w} = (w_0, w_1, ..., w_{r-1})$ as in 2.1. Note that $w \in Z'_{\mathscr{L}}$ and so $\mathbf{w} \in \mathbf{Z}_{\mathscr{L}}$ in the notation of 1.4. It follows from Proposition 1.11 that we have

(2.2.1) All the irreducible constituents in ${}^pH^i(K_{\mathbf{w}})$ are contained in $\hat{G}_{\mathscr{L}}$.

We now assume that A_0 is $F\dot{w}$-stable and fix an isomorphism φ_0: $(F\dot{w})^* A_0 \xrightarrow{\sim} A_0$. Following [L4, 5.5] we shall construct an isomorphism $f: F^*K_{\mathbf{w}} \xrightarrow{\sim} K_{\mathbf{w}}$. The isomorphism φ_0 induces an isomorphism $\tilde{p}(\varphi_0)$: $(\hat{F})^* \tilde{p}A_0 \xrightarrow{\sim} \tilde{p}A_0$ on $\hat{V}'$ by (2.1.3). There exists a perverse sheaf K_1 on V such that $\tilde{p}A_0 = \tilde{\pi}''K_1$. Then by (2.1.3), we see that $\tilde{p}(\varphi_0)$: $\tilde{\pi}''\bar{F}^*K_1 \xrightarrow{\sim} \tilde{\pi}''K_1$. Since π'' is smooth with connected fibres, one can find a unique f_1: $\bar{F}^*K_1 \xrightarrow{\sim} K_1$ such that $\tilde{\pi}''(f_1) = \tilde{p}(\varphi_0)$. Since $\bar{F}$ is proper by 2.1, f_1 gives rise to an isomorphism f_1': $K_1 \xrightarrow{\sim} \bar{F}_!K_1$. Hence we have $\pi_!(f_1')$: $\pi_!K_1 \xrightarrow{\sim} \pi_!\bar{F}_!K_1$. But by (2.1.3) $\pi_!\bar{F}_!K_1$ is naturally isomorphic to $F_!\pi_!K_1$. Since $K_{\mathbf{w}} = \pi_!K_1$ by definition, and since F is proper, $\pi_!(f_1')$ gives rise to an isomorphism $f: F^*K_{\mathbf{w}} \xrightarrow{\sim} K_{\mathbf{w}}$.

2.3. Let F_0^s be a power of F_0 such that $F_0^s = F^s = (F\dot{w})^s$. Then φ_0: $(F\dot{w})^* A_0 \xrightarrow{\sim} A_0$ induces an isomorphism $\varphi_0^{(s)}$: $(F_0^s)^* A_0 \xrightarrow{\sim} A_0$. Since the varieties $\hat{V}$, $\hat{V}'$ have natural $\mathbf{F}_{q^s}$-structures given by the Frobenius map F_0^s, $K_{\mathbf{w}}$ has a natural $\mathbf{F}_{q^s}$-structure given by an isomorphism f_0: $(F_0^s)^* K_{\mathbf{w}} \xrightarrow{\sim} K_{\mathbf{w}}$ induced from $\varphi_0^{(s)}$. We consider the category of mixed complexes on G_0 (here G_0 means a split $\mathbf{F}_{q^s}$-form of G). Then $K_{\mathbf{w}}$ may be regarded as a mixed complex on G_0. Since F commutes with F_0^s, $F^*K_{\mathbf{w}}$ is also a mixed complex on G_0. Thus, ${}^pH^i(K_{\mathbf{w}})$ and ${}^pH^i(F^*K_{\mathbf{w}})$ are mixed perverse sheaves on G_0, and have natural weight filtrations whose successive quotients are pure perverse sheaves of weight j, which we denote by ${}^pH_j^i(K_{\mathbf{w}})$ and ${}^pH_j^i(F^*K_{\mathbf{w}})$. Now φ_0 is a morphism in the derived category of mixed perverse sheaves on L_0 (a split $\mathbf{F}_{q^s}$-form of L) since

$$\varphi_0 \circ (Fw)^* (\varphi_0^{(s)}) = \varphi_0^{(s)} \circ (F_0^s)^* (\varphi_0)$$

(cf. [BDD, 5.1.2]). It follows that f: $F^*K_{\mathbf{w}} \xrightarrow{\sim} K_{\mathbf{w}}$ turns out to be a morphism of mixed complexes on G_0. Hence the induced map $\hat{f}$: ${}^pH^i(F^*K_{\mathbf{w}}) \xrightarrow{\sim} {}^pH^i(K_{\mathbf{w}})$ is a morphism in the category of mixed perverse sheaves on G_0, and so, by [BDD, 5.3.5] it preserves the weigth filtrations of ${}^pH^i(F^*K_{\mathbf{w}})$ and ${}^pH^i(K_{\mathbf{w}})$. We shall show that

(2.3.1) $F^{*p}H^i(K_{\mathbf{w}})$ is a mixed perverse sheaf on G and f induces an isomorphism $F^{*p}H^i(K_{\mathbf{w}}) \xrightarrow{\sim} {}^pH^i(K_{\mathbf{w}})$ as mixed perverse sheaves.

First we note, that ${}^pH^i_j(K_{\mathbf{w}})$ is a semisimple perverse sheaf and, by (2.2.1), that its all irreducuble components are in $\hat{G}_{\mathscr{L}}$. Since F^* preserves the set of character sheaves on G, $F^{*p}H^i_j(K_{\mathbf{w}})$ is also a semisimple perverse sheaf. We consider the natural filtration $W_0 \subset W_1 \subset \cdots$ of ${}^pH^i(K_{\mathbf{w}})$ so that $W_j/W_{j-1} = {}^pH^i_j(K_{\mathbf{w}})$. Then $(F^*W_{j-1}, F^*W_j, F^{*p}H^i_j(K_{\mathbf{w}}))$ is a distinguished triangle in $\mathscr{D}G$, and applying the functor ${}^pH^0$ to it, we get a long exact sequence of perverse cohomologies, where ${}^pH^k(F^{*p}H^i_j(K_{\mathbf{w}})) = 0$ for any $k \neq 0$. It follows, by induction on j, that ${}^pH^k(F^*W_j) = 0$ for any $k \neq 0$. This implies that F^*W_j and so $F^{*p}H^i(K_{\mathbf{w}})$ is a perverse sheaf by [BBD, 1.3.7]. Since F commutes with F_0^*, it is a mixed perverse sheaf on G_0. Now, by [BBD, 1.3.17], ${}^pH^i(F^*K_{\mathbf{w}})$ is naturally isomorphic to $F^{*p}H^i(K_{\mathbf{w}})$, in the category of mixed perverse sheaves. Thus by composing with $\hat{f}$, we get (2.3.1).

Next we show

(2.3.2) F^* maps the natural filtration of ${}^pH^i(K_{\mathbf{w}})$ to that of $F^{*p}H^i(K_{\mathbf{w}})$. Hence f induces an isomorphism $\tilde{f}$: $F^{*p}H^i_j(K_{\mathbf{w}}) \xrightarrow{\sim} {}^pH^i_j(K_{\mathbf{w}})$ for each j.

In fact, the isomorphism $F^{*p}H^i(K_{\mathbf{w}}) \xrightarrow{\sim} {}^pH^i(K_{\mathbf{w}})$ preserves weigth filtrations, and so it induces an isomorphism on successive quotients. On the other hand, $F^*(W_0) \subset F^*(W_1) \subset \cdots$ gives a filtration of $F^{*p}H^i(K_{\mathbf{w}})$. Since F^* preserves $\mathscr{D}G^{\leq i}$ for any i, this filtration necessarily coincides with the natural filtration of $F^{*p}H^i(K_{\mathbf{w}})$. This proves (2.3.2).

2.4. The complex ${}^pH^i_j(K_{\mathbf{w}})$ is decomposed as

$$ {}^pH^i_j(K_{\mathbf{w}}) \simeq \bigoplus_A (A \otimes V_{A,i,j}), $$

where A runs over the set $\hat{G}_{\mathscr{L}}$ (cf. (2.2.1)), and $V_{A,i,j}$ are finite dimensional $\bar{\mathbf{Q}}_l$-vector spaces. We identify $V_{A,i,j}$ with $\operatorname{Hom}_{\mathscr{M}G}(A, {}^pH^i_j(K_{\mathbf{w}}))$. The complex $F^{*p}H^i_j(K_{\mathbf{w}})$ is decomposed similarly,

$$ F^{*p}H^i_j(K_{\mathbf{w}}) \simeq \bigoplus_A (F^*A \otimes V'_{F^*A,i,j}), $$

where $V'_{F^*A,i,j} = \operatorname{Hom}_{\mathscr{M}G}(F^*A, F^{*p}H^i_j(K_{\mathbf{w}}))$. We may identify $V'_{F^*A,i,j}$ with $V_{A,i,j}$ by the natural isomorphism

$$ \operatorname{Hom}_{\mathscr{M}G}(A, {}^pH^i_j(K_{\mathbf{w}})) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{M}G}(F^*A, F^{*p}H^i_j(K_{\mathbf{w}})) $$

via $\varphi \mapsto F^*(\varphi)$. Suppose now that A is F-stable and an isomorphism ϕ_A: $F^*A \xrightarrow{\sim} A$ is given. Then under the above identification, the isomorphism in (2.3.2) is decomposed into isomorphisms

$$ \phi_A \otimes \Psi_{A,i,j}: F^*A \otimes V_{A,i,j} \xrightarrow{\sim} A \otimes V_{A,i,j}. $$

where $\Psi_{A,i,j}: V_{A,i,j} \to V_{A,i,j}$ is a linear map defined by $\Psi_{A,i,j}(\varphi) = \bar{f} \circ F^*(\varphi) \circ \phi_A^{-1}$.

2.5. The similar procedure works also for the isomorphism f_0: $(F_0^s)^* K_{\mathbf{w}} \xrightarrow{\sim} K_{\mathbf{w}}$, replacing F by $F_0^s = F^s$. Namely, F_0 induces an isomorphism $\bar{f}_0$: $(F_0^s)^* {}^pH_j^i(K_{\mathbf{w}}) \xrightarrow{\sim} {}^pH_j^i(K_{\mathbf{w}})$ and it is decomposed, on each A-isotypic component, as

$$\phi_A^{(s)} \otimes \psi_{A,i,j}: (F^s)^* A \otimes V_{A,i,j} \xrightarrow{\sim} A \otimes V_{A,i,j},$$

where $\phi_A^{(s)}: (F^s)^* A \xrightarrow{\sim} A$ is the ismorphism induced from ϕ_A: $F^*A \xrightarrow{\sim} A$, and $\psi_{A,i,j}$ is defined in a similar way as before, i.e., $\psi_{A,i,j}(\varphi) = \bar{f}_0 \circ (F_0^s)^* (\varphi) \circ (\phi_A^{(s)})^{-1}$ for $\varphi \in \operatorname{Hom}_{\mathscr{M}G}(A, {}^pH_j^i(K_{\mathbf{w}}))$.

If F_0^s is replaced by F_0^{sm} for a positive integer m, we have a natural isomorphism $\bar{f}_0^{(m)}$: $(F_0^{ms})^* {}^pH_j^i(K_{\mathbf{w}}) \xrightarrow{\sim} {}^pH_j^i(K_{\mathbf{w}})$, and $\phi_A^{(ms)}$: $(F^{ms})^* A \xrightarrow{\sim} A$. Then $\bar{f}_0^{(m)}$ is described on each A-isotypic component of ${}^pH_j^i(K_{\mathbf{w}})$ as

$$\phi_A^{(ms)} \otimes \psi_{A,i,j}^m: (F^{ms})^* A \otimes V_{A,i,j} \xrightarrow{\sim} A \otimes V_{A,i,j}.$$

We now consider an isomorphism f': $(F^{ms+1})^* K_{\mathbf{w}} \xrightarrow{\sim} K_{\mathbf{w}}$ defined by

$$f' = f_0^{(m)} \circ (F^{ms})^* (f).$$

Then f' coincides with the isomorphism induced from $\varphi_0^{(ms+1)}$: $(F^{ms+1}\dot{w})^* A_0 \xrightarrow{\sim} A_0$ in a similar way as f, but replacing F by F^{ms+1}. The isomorphism f' induces an isomorphism $\bar{f}'$: $(F^{ms+1})^* {}^pH_j^i(K_{\mathbf{w}}) \xrightarrow{\sim} {}^pH_j^i(K_{\mathbf{w}})$, and it is described as

$$\phi_A^{(ms+1)} \otimes (\psi_{A,i,j}^m \circ \Psi_{A,i,j}): (F^{ms+1})^* A \otimes V_{A,i,j} \to A \otimes V_{A,i,j}. \quad (2.5.1)$$

Note that the isomorphisms $\psi_{A,i,j}$ and $\Psi_{A,i,j}$ on $V_{A,i,j}$ commute each other.

2.6. We define, for a positive integer c, a set $\mathscr{M}_c$ as the set of $m \in \mathbf{N}$ such that $m \equiv 1 (\operatorname{mod} c)$. We fix an integer s such that it satisfies the condition for s in 2.3, and that $(F^s)^* A \simeq A$ for any $A \in \hat{G}_{\mathscr{L}}$. For each $m \in \mathscr{M}_s$, we consider the characteristic function $\chi_{K_{\mathbf{w}}, f'}$: $G^{F^m} \to \bar{\mathbf{Q}}_l$. We shall express it in terms of characteristic functions $\chi_A^{(m)} = \chi_{A, \phi_A^{(m)}}$ of F-stable character sheaves A. By using the spectral sequence of the stalks at $g \in G$,

$$E_2^{a,b} = \mathscr{H}_g^a({}^pH^b(K_{\mathbf{w}})) \Rightarrow \mathscr{H}_g^{a+b}(K_{\mathbf{w}}),$$

we have

$$\chi_{K_{\mathbf{w}}, f'}(g) = \sum_{a,b} (-1)^{a+b} \operatorname{Tr}(f', \mathscr{H}_g^a({}^pH^bK_{\mathbf{w}}))$$

for each $g \in G^{F^m}$. The right hand side can be computed by using the natural filtration of ${}^pH^h(K_{\mathbf{w}})$, and on each subquotient ${}^pH^h_i(K_{\mathbf{w}})$, f' is described as in (2.5.1). Note that by our assumption on s, only F-stable character sheaves enter in the computation. Thus we have the following lemma which is analogous to [S3, Lemma 2.10].

LEMMA 2.7. *For each F-stable A in $\hat{G}_{\mathscr{S}}$, there exist finite subsets $\{\xi_i \mid i \in I_A\}$ and $\{c_i \mid i \in I_A\}$ of $\bar{\mathbf{Q}}_l$ satisfying the following formula: for each $m \in \mathscr{M}_s$ (s: as in 2.6), let f': $(F^m)^* K_{\mathbf{w}} \xrightarrow{\sim} K_{\mathbf{w}}$ be as before. Then*

$$\chi_{K_{\mathbf{w}}, f'} = \sum_A \Big(\sum_{i \in I_A} c_i \xi_i^m \Big) \chi_A^{(m)},$$

where A runs over all the elements in $\hat{G}_{\mathscr{S}}$ such that $F^*A \simeq A$.

2.8. Following [L4, 5], we shall express the function $\chi_{K_{\mathbf{w}}, f}$ in terms of the characteristic function χ_{A_0, φ_0} of A_0. First we remark that for any $g \in G^F$ the restriction of $\bar{F}$ on $\pi^{-1}(g)$ is a Frobenius map on it with respect to some $\mathbf{F}_q$-structure. In fact, let V_g be the union of $\pi^{-1}(g')$ for g' in the σ-orbit of g. Then V_g is $\bar{F}$-stable, and so it is enough to show that the restriction of $\bar{F}$ on V_g is a Frobenius map. Now V_g is τ and F_0-stable, and $\bar{F} = \tau F_0$ on V_g. Since τ commutes with F_0, it is enough to show that τ is of finite order on V_g. But for $(g, h_0 P, ..., h_r P) \in V$ we have

$$\tau^{r+1}(g, h_0 P, ..., h_r P) = (\sigma^{r+1}(g), \sigma^{r+1}(g^{-1} h_0) P, ..., \sigma^{r+1}(g^{-1} h_r) P).$$

Hence, for any i, we have

$$\begin{aligned} &\tau^{i(r+1)}(g, h_0 P, ..., h_r P) \\ &\quad = (\sigma^{i(r+1)}(g), \sigma^{i(r+1)}(g^{-i}) \sigma^{i(r+1)}(h_0) P, ..., \sigma^{i(r+1)}(g^{-i}) \sigma^{i(r+1)}(h_r) P). \end{aligned}$$

This implies that τ is of finite order on V_g. (Note that this result is also contained in [L4, Lemma 5.8]).

Now, for each $g \in G^F$,

$$\begin{aligned} \chi_{K_{\mathbf{w}}, f}(g) &= \sum_i (-1)^i \operatorname{Tr}(f, \mathscr{H}^i_g(K_{\mathbf{w}})) \\ &= \sum_i (-1)^i \operatorname{Tr}(\Psi, \mathbf{H}^i_c(\pi^{-1}(g), K_1)), \end{aligned}$$

where Ψ: $\mathbf{H}^i_c(\pi^{-1}(g), K_1) \xrightarrow{\sim} \mathbf{H}^i_c(\pi^{-1}(g), K_1)$ is the map induced from the isomorphism $K_1 \xrightarrow{\sim} \bar{F}_! K_1$, restricted on $\pi^{-1}(g)$. Since $\bar{F}$ is a Frobenius map on $\pi^{-1}(g)$, the last sum can be computed using the Grothendieck trace formula. Then by the similar argument as in the proof of Proposition 5.7 in [L4], we have the following formula.

LEMMA 2.9 (cf. [L4, Prop. 5.7]). *For each $g \in G$,*

$$\chi_{K_{\mathbf{w}},f}(g) = (-1)^{d_{\mathbf{w}}} |L^{F\dot{w}}|^{-1} \sum_{h \in \Theta_g} \chi_{A_0, \varphi_0}(\mathrm{pr}_L(F^r(h)^{-1} gh)),$$

where pr_L is the projection from $P = LU_P$ to L, and

$$d_{\mathbf{w}} = \dim G + \sum_{i=0}^{r-1} \dim U^-_{w_i} - r \cdot \dim_L \qquad \textit{see } (1.11.6),$$

$$= \dim G + r \cdot \dim U^-_w - r \cdot \dim L.$$

Moreover,

$$\Theta_g = \{hU_P \in G/U_P \mid h^{-1}F(h) \in U_P F(\dot{w})\, U_P,\ F^r(h)^{-1} gh \in L^{F\dot{w}} U_P\}.$$

2.10. As in Lemma 2.12 in [S3], it is possible to describe the right hand side of Lemma 2.9 in terms of certain induced representations of G^{F^r} by making use of Shintani descent. For this, we review some notations on Shintani descent (cf. [S1, 1.12]). Suppose we are in a setup in 2.1. Let π be an irreducible representation of L^{F^r} which is stable by $F\dot{w}$. We denote by θ the restriction of F on L^{F^r}. We fix an extension $\tilde{\pi}$ of π to the semidirect product $L^{F^r}\langle \theta\dot{w}\rangle$, where $\theta\dot{w}$ is the restriction of $F \circ \mathrm{ad}\, \dot{w}$ on L^{F^r}. We lift π to a representation of P^{F^r} in a natural way and denote it also by π. Fixing a representation space V of π, we consider the space $\mathscr{P}$ of all functions f: $G^{F^r} \to V$ with G^{F^r}-module structure by $(gf)(x) = f(xg)$, $(g, x \in G^{F^r}, f \in \mathscr{P})$. Let $\mathscr{P}_\pi$ be the subspace of $\mathscr{P}$ defined by

$$\mathscr{P}_\pi = \{f \in \mathscr{P} \mid f(pg) = \pi(p) f(g) \text{ for } p \in P^{F^r}, g \in G^{F^r}\}.$$

Then $\mathscr{P}_\pi$ is a G^{F^r}-submodule of $\mathscr{P}$ realizing $\mathrm{Ind}_{P^{F^r}}^{G^{F^r}} \pi$. We define a linear map $\tau_{\pi, \dot{w}}$: $\mathscr{P} \to \mathscr{P}$ by

$$\tau_{\pi, \dot{w}}(f)(x) = |U_P^{F^r}|^{-1} \sum_{y \in U_P^{F^r}} f(\dot{w}^{-1} yx).$$

We define also, F: $\mathscr{P} \to \mathscr{P}$ by $F(f)(x) = f(F^{-1}(x))$, and $\tilde{\pi}(\theta\dot{w})$: $\mathscr{P} \to \mathscr{P}$ by $\tilde{\pi}(\theta\dot{w})(f)(x) = \tilde{\pi}(\theta\dot{w})(f(x))$, $(f \in \mathscr{P}, x \in G^{F^r})$. Then it is easily verified that a linear map $\tilde{\pi}(\theta\dot{w})\, F\tau_{\pi, \dot{w}}$: $\mathscr{P} \to \mathscr{P}$ leaves $\mathscr{P}_\pi$ invariant. Now let $C(G^{F^r}/\sim_F)$ (resp. $C(L^{F^r}/\sim_{F\dot{w}})$) be the space of $\bar{\mathbf{Q}}_l$-valued functions on the set $G^{F^r}/\sim_F$ (resp. $L^{F^r}/\sim_{F\dot{w}}$), (see [S3, 1.16]). The restriction $\tilde{\pi} \mid L^{F^r}\theta\dot{w}$ of $\tilde{\pi}$ may be regarded as an element in $C(L^{F^r}/\sim_{F\dot{w}})$. We define a map $a_{F\dot{w}}$: $C(L^{F^r}/\sim_{F\dot{w}}) \to C(G^{F^r}/\sim_F)$ by

$$a_{F\dot{w}}(\tilde{\pi} \mid L^{F^r}\theta\dot{w})(x) = \mathrm{Tr}(x\tilde{\pi}(\theta\dot{w})\, F\tau_{\pi, \dot{w}}, \mathscr{P}_\pi),$$

for each $F\dot{w}$-stable irreducible representation π, and by extending it linearly to the whole space. It is easily verified that for any $f \in C(L^{F'}/\sim_{F\dot{w}})$, $\hat{g} \in G^{F'}$, we have

$$a_{F\dot{w}}(f)(\hat{g}) = |L^{F'}|^{-1} \sum_{l \in L^{F'}} c_{\hat{g},l} f(l\theta\dot{w}) \tag{2.10.1}$$

where

$$c_{\hat{g},l} = \#\{h \in (G/U_P)^{F'} \mid l^{-1}h\hat{g}F(h^{-1}) \in U_P F(\dot{w})\, U_P\}.$$

We now consider the Shintani descent

$$N^*_{F'/F\dot{w}}: C(L^{F\dot{w}}/\sim) \to C(L^{F'}/\sim_{F\dot{w}})$$

(see [S3, 1.16]). The following proposition is an extended version of Lemma 2.12 in [S3], and also an analogue of the Shintani descent identity (cf. [S1, Prop. 1.13]).

PROPOSITION 2.11.

$$\chi_{K_{w},f}(g) = (-1)^{d_w} a_{F\dot{w}}(N^*_{F'/F\dot{w}}(\chi_{A_0,\varphi_0}))(\hat{g}),$$

where $g \in G^F$, *and* $\hat{g} \in G^{F'}$ *are related by* $N_{F'/F}(\hat{g}) = g$.

Proof. By Lemma 2.9, $\chi_{K_w,f}(g)$ is written as

$$\chi_{K_w,f}(g) = (-1)^{d_w} |L^{F\dot{w}}|^{-1} \sum_{l \in L^{F\dot{w}}} b_{g,l}\chi_{A_0,\varphi_0}(l),$$

where

$$b_{g,l} = \#\{hU_P \in G/U_P \mid h^{-1}F(h) \in U_P F(\dot{w})\, U_P,\ l^{-1}F'(h)^{-1}gh \in U_P\}.$$

By using Lang's theorem for the Frobenius map $(\operatorname{ad} l^{-1}) \circ F$ on U_P, we see that $b_{g,l} = |U_P^{F'}|^{-1} |B_{g,l}|$, where

$$B_{g,l} = \{h \in G \mid h^{-1}F(h) \in U_P F(\dot{w})\, U_P,\ F'(h)^{-1}gh = l\}.$$

We choose $\alpha \in G$, $\beta \in L$ such that $g = F'(\alpha)\,\alpha^{-1}$, $l = F'(\beta)\,\beta^{-1}$ and put $\hat{g} = \alpha^{-1}F(\alpha) \in G^{F'}$, $\hat{l} = \beta^{-1}F(\dot{w}\beta\dot{w}^{-1}) \in L^{F'}$. Let $h' = \alpha^{-1}h\beta$ for $h \in B_{g,l}$. Then $h' \in G^{F'}$ and satisfies the relation $\hat{l}^{-1}h'^{-1}\hat{g}F(h') \in U_P F(\dot{w})\, U_P$. It follows that $b_{g,l} = c_{\hat{g},\hat{l}}$ in the notation of (2.10.1). Now the proposition follows from (2.10.1) by making use of a simple property of Shintani descent (see e.g. [L1, Prop. 2.10]).

Replacing F by F^m for $m \in \mathcal{N}_s$, and combining Proposition 2.11 with Lemma 2.7, we have the following.

COROLLARY 2.12. *For each F-stable A in $\hat{G}_{\mathscr{L}}$, there exist finite subsets $\{\xi_i \mid i \in I_A\}$ and $\{c_i \mid i \in I_A\}$ of $\bar{\mathbf{Q}}_l$ satisfying the following. For each $m \in \mathscr{M}_s$,*

$$(-1)^{d_w} N^{*\,-1}_{F^{mr}/F^m} \circ a_{F^m w} \circ N^*_{F^{mr}/F^m w}(\chi^{(m)}_{A_0,\varphi_0}) = \sum_A \Big(\sum_{i \in I_A} c_i \xi_i^m \Big) \chi_A^{(m)}, \qquad (2.12.1)$$

where s is as in 2.6 and A runs over all the elements in $\hat{G}_{\mathscr{L}}$ such that $F^*A \simeq A$.

3. LUSZTIG'S CONJECTURE FOR CLASSICAL GROUPS

3.1. From now on, throughout the remainder of the paper, we assume that the center of G is connected. We preserve the notation in 2.1. Let G^* be the dual group of G, and T^* a maximal torus of G^* dual to T. If we fix an isomorphism $\iota: k^* \xrightarrow{\sim} \mathbf{Q}'/\mathbf{Z}$, the map $\nu \otimes (1/n) \mapsto \nu(\iota^{-1}(1/n))$ gives rise to an isomorphism $\mathrm{Hom}(k^*, T^*) \otimes \mathbf{Q}'/\mathbf{Z} \xrightarrow{\sim} T^*$. By the definition of the dual torus, we have an isomorphism

$$\mathrm{Hom}(k^*, T^*) \simeq \mathrm{Hom}(T, k^*). \qquad (3.1.1)$$

Thus, in view of (1.4.1), we get an isomorphism

$$T^* \simeq \mathscr{S}(T). \qquad (3.1.2)$$

Let $W^* = N_{G^*}(T^*)/T^*$ be the Weyl group of G^*. W^* may be identified with $W = N_G(T)/T$, in such a way compatible with the isomorphism (3.1.2). Note that the action of F on $\mathscr{S}(T)$ corresponds to the action of F^{-1} on T^*, since the isomorphism (3.1.1) satisfies the property that $F \circ \nu$ corresponds to $\lambda \circ F$ if $\nu \in \mathrm{Hom}(k^*, T^*)$ corresponds to $\lambda \in \mathrm{Hom}(T, k^*)$. For each $s \in T^*$ such that the conjugacy class $\{s\}$ in G^* is F-stable, the subsets W_s and Z_s of W^* are defined as in [S3, 5.1]. We fix a local system $\mathscr{L} \in \mathscr{S}(T)$ corresponding to s under (3.1.2). Then W_s may be identified with $W'_{\mathscr{L}} = W_{\mathscr{L}}$ as G has connected center, and Z_s may be identified with $Z'_{\mathscr{L}} = Z_{\mathscr{L}}$. Now $Z_{\mathscr{L}}$ is written as $z_1 W_{\mathscr{L}}$ as in 1.5, and we have an automorphism $\gamma = \gamma_s = z_1^{-1}F: W_s \to W_s$. (Note that in [S3, 5.1], z_1 is written as w_0). As explained in [S3, 5.2], we have two types of parameter sets, one is $X(W_s, \gamma)$, which is used for describing almost characters attached to the set $\mathscr{E}(G^F, \{s\})$, and the other is $\bar{X}(W_s)^{\gamma}$, used for describing F-stable character sheaves in $\hat{G}_{\mathscr{L}}$. We have a natural surjection $X(W_s, \gamma) \to \bar{X}(W_s)^{\gamma}$, which is denoted by $x \mapsto \bar{x}$. We denote by R_x the almost character of G^F corresponding to $x \in X(W_s, \gamma)$, and denote by $A = A_x$ the F-stable character sheaf corresponding to $\bar{x} \in \bar{X}(W_s)^{\gamma}$. We fix an isomorphism ϕ_A: $F^*A \xrightarrow{\sim} A$ as in [S3, 1.4], and denote by $\chi_A = \chi_{A,\phi_A}$ the corresponding characteristic function. As in [S3, 1.13] an algebraic number ξ_A of absolute value 1 is attached to each $A = A_x \in \hat{G}$, which we denote by ξ_x. Recall that Lusztig's conjecture connects the almost characters to chracteristic functions of character sheaves, (cf. [S3, 5.6]). We shall state the following theorem, which is slightly weaker than the conjecture itself.

THEOREM 3.2. *Assume that the center of G is connected. Assume also that G is a quasi simple group of classical type. Then for each $x \in X(W_s, \gamma)$, there exists an algebraic number ε_x of absolute value 1 such that*

$$R_x = \varepsilon_x \xi_x \chi_{A_x}. \tag{3.2.1}$$

Except for the cases where G is of type ${}^2D_n$ or ${}^3D_4$, ε_x may be taken to be ± 1.

Remark 3.3. In [S3, Th. 5.7], the theorem was proved for the groups of type A_n, B_n or C_n. (The statement in Theorem 5.7 in [S3] is inadequate. What was really proved there is of the above form. See Lemma 5.19 in [S3]). Contrast to the method employed there (which requires more delicate investigation for the case of type D_n and so, that case remained to be solved), the argument in this paper works as well to the case of type D_n. So, in the following proof, we discuss the case of classical groups in a simultaneous way.

3.4. In the proof below, we are only concerned with the set $\bar{X}(W_s)^\gamma$ and not with $X(W_s, \gamma)$. So, we express the elements $\bar{x}$ in $\bar{X}(W_s)^\gamma$ simply by x (by removing the bar), and denote the corresponding objects such as $A_{\bar{x}}$, $\xi_{\bar{x}}$ by A_x, ξ_x, respectively. Moreover, for each $x \in \bar{X}(W_s)^\gamma$ we choose an element $x' \in X(W_s, \gamma)$ in the inverse image of x, and denote the corresponding almost character $R_{x'}$ by R_x. Note that R_x depends only on x up to a scalar multiple.

We shall prove the theorem by induction on the rank of G. So we assume that the theorem was proved for any proper F-stable Levi subgroup of a parabolic subgroup of G having the same type as G. We fix $s \in T^*$ and the corresponding $\mathscr{L} \in \mathscr{S}(T)$ as in 3.1. Let L be an F-stable Levi subgroup containing T as in 2.1. We assume that $\hat{L}_{\mathscr{L}}$ contains a cuspidal character sheaf A_0, where $A_0 = \mathrm{IC}(\bar{\Sigma}, \mathscr{E})[\dim \Sigma]$ for a cuspidal pair $(\Sigma, \mathscr{E})$ on L. Then either L is a maximal torus or L has the same type as G. We consider the complex $K = \mathrm{ind}_P^G A_0$ on G. By Lemma 5.9 in [S3] the endomorphism algebra $\mathrm{End}_{\mathscr{M}G}(K)$ is isomorphic to the group algebra $\bar{\mathbf{Q}}_l[\mathscr{W}_{\mathscr{E}}]$ of $\mathscr{W}_{\mathscr{E}}$, where

$$\mathscr{W}_{\mathscr{E}} = \{n \in N_G(L) \mid n\Sigma n^{-1} = \Sigma, \mathrm{ad}(n)^* \mathscr{E} \simeq \mathscr{E}\}/L.$$

$$\mathscr{Z}_{\mathscr{E}} = \{n \in N_G(L) \mid F(n\Sigma n^{-1}) = \Sigma, \mathrm{ad}(n)^* F^*\mathscr{E} \simeq \mathscr{E}\}/L.$$

If we choose a positive integer r large enough, the set $\mathscr{E}(L^{F^r}, \{s\})$ contains a cuspidal representation δ of L^{F^r}. We define

$$W_\delta = \{w \in N_{W}(W_L) \mid wB_L w^{-1} = B_L, {}^w\delta \simeq \delta\},$$
$$Z_\delta = \{w \in N_{W}(W_L) \mid wB_L w^{-1} = B_L, {}^{Fw}\delta \simeq \delta\}.$$

Since $\hat{L}_{\mathscr{S}}$ contains a unique cuspidal character sheaf, we have $W_\delta \simeq \mathscr{W}_{\mathscr{E}}$ and $Z_\delta \simeq \mathscr{Z}_{\mathscr{E}}$ (cf. (5.16.1) in [S3]). Moreover there exists $w_1 \in Z_\delta$ such that $Z_\delta = w_1 W_\delta$ and that $\gamma_1 = Fw_1$: $W_\delta \to W_\delta$ gives rise to an automorphism of Coxeter groups (cf. [S3, 5.15]). Let L_{w_1} be an F-stable Levi subgroup twisted by $F(w_1)$, i.e., $L_{w_1} = \alpha L \alpha^{-1}$ for $\alpha \in G$ such that $\alpha^{-1}F(\alpha) = F(\dot{w}_1)$ for a representative $\dot{w}_1 \in N_G(L)$ of w_1. Then by ad α: $L \simeq L_{w_1}$, $\mathrm{ad}(\alpha^{-1})^* A_0$ turns out to be an F-stable cuspidal character sheaf on L_{w_1}, which we denote also by A_0. If we fix an isomorphism φ_0: $(F\dot{w}_1)^* \mathscr{E} \simeq \mathscr{E}$, φ_0 induces an isomorphism $\varphi_0^{w_1}$: $F^*A_0 \simeq A_0$ on L_{w_1}. We may assume that $\varphi_0^{w_1}$ coincides with ϕ_{A_0}. Then by induction hypothesis, we have

$$R_0^{L_{w_1}} = \varepsilon_0 \xi_{A_0} \chi_{A_0} \tag{3.4.1}$$

where $R_0^{L_{w_1}}$ is the almost character of $L_{w_1}^F$ corresponding to $A_0 \in \hat{L}_{w_1}$. Let $\tilde{W}_\delta$ be the semidirect product of W_δ with the cyclic group generated by γ_1. We denote by $(W_\delta)^\wedge_{\mathrm{ex}}$ the set of irreducible representations of W_δ which is extendable to $\tilde{W}_\delta$. For each $E \in (W_\delta)^\wedge_{\mathrm{ex}}$, there exists $x_E \in \bar{X}(W_s)^\gamma$ such that $A_E = A_{x_E}$ is an F-stable character sheaf in $\hat{G}_{\mathscr{S}}$ which is a simple component of K corresponding to $E \in \mathrm{End}_{\mathscr{M}G}(K)^\wedge$. The we have the following.

(3.4.2) Suppose q is large enough so that Theorem 1.1 in [S3] can be applied. Then

$$R_{x_E} = (-1)^{\dim \Sigma} \varepsilon_0 \xi_{A_E} \chi_{A_E}$$

for any $E \in (W_\delta)^\wedge_{\mathrm{ex}}$.

In fact, (3.4.2) can be proved using the similar argument given in the proof of (5.20.1) in [S3]. We note that (5.20.2) in [*loc. cit.*] holds also for the case of type D_n. This is shown as follows. If $p = 2$, G has at most one cuspidal character sheaf ([L3, V, 22.2]), and the argument in [S3] can be applied. So, we assume that p is odd. In this case $L' = L/Z^0(L)$ is isomorphic to PSO_{2n} and $(L')^* \simeq \mathrm{Spin}_{2n}$. As described in [L3, V, 23.2], the set of cuspidal character sheaves in L' are partitioned into subsets according to their supports, the closure of conjugacy classes g in L'. If $\{A'_i \mid i \in I_g\}$ is such a subset, the cardinality $|I_g| = 1$, 2 or 4. The class of g is F-stable and there exists A'_{i_0} which is stable by F, and any other A'_i $(i \in I_g)$ are obtained by tensoring tame local systems on L' corresponding to elements in the center of $(L')^*$. This holds for any Frobenius map. Thus when we express a cuspidal character sheaf $A_0 \in \hat{L}$ as $A_0 = \mathscr{E}_0 \otimes \tilde{p} A'_0$ (cf. (5.20.2) in [S3]), we may assume, by replacing $\mathscr{E}_0$ by a suitable local system on L if necessary, that A'_0 is F-stable. From the above remark, actually A'_0 is $F\dot{w}_1\dot{w}$-stable for any $w \in W_\delta$. Now the remaining part of the proof of (5.20.2) is valid for our case without change.

3.5. Our next aim is to remove the assumption on q in (3.4.2). But before doing it, we need some preliminary. Recall, by [L1], that for each irreducible character ρ of G^F, a root of unity $\lambda_\rho \in \bar{\mathbf{Q}}_l^*$ is attached; λ_ρ is characterized by the following property, (cf. [S3, 2.15]). Let (λ, n) be the pair corresponding to $\mathscr{L} \in \mathscr{S}(T)$ under (1.4.1), and let $\mathscr{F}^\lambda_{\dot{w},n}$ be the local system on the variety X_w associated to w, (see [L1, 2.3]). Let d be as in 1.4, and let a be the smallest integer such that F^a is a power of F_0^d. Then F^a acts on $H^i_c(X_w, \mathscr{F}^\lambda_{\dot{w},n})$ and any constituent ρ in it is contained in the generalized eigenspace of F^a with the eigenvalues of the form $\lambda_\rho q^{ja/2}$ for some integer $j \leqslant i$. Note that a priori λ_ρ depends on the choice of F since X_w and ρ does. However we can show that the order of λ_ρ for various ρ is bounded if F is replaced by any power F^m, i.e., we have the following lemma. (For later use in Section 4, we show it for general G).

LEMMA 3.6. *Let G be a reductive group with connected center. Assume that G is simple modulo center. Then there exists an integer N depending only on the Dynkin diagram of G such that $\lambda_\rho^N = 1$ for any $\rho \in \hat{G}^{F'}$ with respect to any Frobenius map F' on G.*

Proof. The determination of λ_ρ is reduced to the case where ρ is cuspidal (cf. [L1, 6.4]). Moreover if ρ is unipotent with split Frobenius F, λ_ρ is explicitly determined in [L1, 11]. The case where ρ is unipotent with non-split Frobenius F is discussed in Remark 2.15 in [S3]. The determination of λ_ρ in the case where ρ belongs to $\mathscr{E}(G^F, \{s\})$ with $s \in Z(G^*)^F$ is reduced to the case where ρ is unipotent by Lemma 1.13 in [A3]. Thus we consider the case where ρ belongs to $\mathscr{E}(G^F, \{s\})$ and $s \notin Z(G^*)$. It is enough to show that all the λ_ρ are contained in some fixed algebraic number field. By the property of λ_ρ stated above, we have the following formula; for each positive integer m,

$$\sum_{i \geqslant 0} (-1)^i \operatorname{Tr}(F^{am}, H^i_c(X_w, \mathscr{F}^\lambda_{\dot{w},n})) = \sum_\rho P_\rho(q^{am/2}) \lambda_\rho^m, \tag{3.6.1}$$

where ρ runs over all the elements in $\mathscr{E}(G^F, \{s\})$ and $P_\rho(u) \in \mathbf{Z}[u]$, a polynomial associated to ρ. The left hand side of (3.6.1) can be computed by making use of Grothendieck's trace formula. Since F^{am} acts trivially on each stalk $(\mathscr{F}^\lambda_{\dot{w},n})_x$ for $x \in X_w^{F^{am}}$, it is equal to $|X_w^{F^{am}}|$. Then we have

$$\sum_\rho P_\rho(q^{am/2}) \lambda_\rho^m = |X_w^{F^{am}}|. \tag{3.6.2}$$

We choose a square root $\sqrt{p}$ of p so that $q^{1/2} \in \mathbf{Q}(\sqrt{p})$. We may assume that all the λ_ρ for non-cuspidal ρ are contained in a fixed cyclotomic field K. If G is of classical type, $\mathscr{E}(G^F, \{s\})$ contains at most one cuspidal character ρ. In that case, (3.6.2) implies that $\lambda_\rho \in K(\sqrt{p})$. If G is of exceptional type, since $s \notin Z(G^*)$, $\mathscr{E}(G^F, \{s\})$ contains at most two cuspidal characters. If it contains only one cuspidal character, the argument as above can be applied. So, we assume that $\mathscr{E}(G^F, \{s\})$ contains two cuspidal characters ρ_1 and ρ_2. Then by Theorem 3.8 and Theorem 4.23 in [L1], one can verify that $P_{\rho_1}(u) = P_{\rho_2}(u)$. It follows that $\lambda_{\rho_1} + \lambda_{\rho_2}, \lambda_{\rho_1}^2 + \lambda_{\rho_2}^2 \in K(\sqrt{p})$, and so $\lambda_{\rho_1}, \lambda_{\rho_2}$ are contained in (at most) fourth degree extension of K. Since λ_{ρ_i} are roots of unity, λ_{ρ_i} are contained in some fixed cyclotomic field. This proves the lemma.

Returning to the case of classical groups, we show the following lemma.

LEMMA 3.7. *The assertion of* (3.4.2) *holds without the assumption of* q.

Proof. We shall prove the lemma by applying a kind of specialization argument as in [S3, Sect. 2]. We choose a coherent lifting $\dot{Z}_\delta$ of Z_δ. We also choose an integer $s > 0$ such that $F_0^s = F^s = (F\dot{w})^s$ for any $\dot{w} \in \dot{Z}_\delta$, and that $(F^s)^* A \simeq A$ for any $A \in \hat{G}_{\mathscr{Y}}$ (cf. 2.6). We consider the set $\mathscr{M}_s$ as in 2.6. It is known (e.g. [S3, Th. 5.13]) that if r is sufficiently divisible, almost characters of G^F are essentially the same as the image of Shintani descent $N_{F^r/F}^{*\,-1}$ of F-stable irreducible characters of G^{F^r}. The number r is determined by the condition on the order of λ_ρ for various $\rho \in \hat{G}^F$. It follows from Lemma 3.6, the least common multiple N of the order of λ_ρ is bounded for any Frobenius map F. Thus one can find r such that Theorem 5.13 in [S3] holds for Shintani descent $N^*_{F^{mr}/F^m}$ on G for any $m \in \mathscr{M}_s$, (and also for Shintani descent on L replacing F by $F\dot{w}_1$). Now by (3.4.1), together with (5.20.2) in [S3], we have for any $m \in \mathscr{M}_s$,

$$(R_0^{L_{w_1 w}})^{(m)} = \varepsilon_0^{(m)} \xi_{A_0}^m \chi_{A_0, \varphi_0^{w_1 w}}^{(m)}, \tag{3.7.1}$$

where $(R_0^{L_{w_1}})^{(m)}$ is the almost character of $L_{w_1 w}^{F^r}$, and $\varepsilon_0^{(m)} = \pm 1$ is a constant depending on m. Now for each $m \in \mathscr{M}_s$, $\mathscr{E}(L^{F^m}, \{s\})$ contains a unique cuspidal character $\delta^{(m)}$ of $L^{F^{mr}}$, and the group $W_{\delta^{(m)}}$ (resp. $Z_{\delta^{(m)}}$) is naturally isomorphic to W_δ (resp. Z_δ) for any $m \in \mathscr{M}_s$. Thus the associated automorphism on $W_{\delta^{(m)}}$ coincide with γ_1 on W_δ for any $m \in \mathscr{M}_s$. Let M be the F-stable subgroup of $N_G(L)$ generated by L and $w \in W_\delta$. Let $\tilde{G}^{F^{mr}}$ be the semidirect product of $G^{F^{mr}}$ with the cyclic group $\langle \theta^m \rangle$ generated by θ^m, where θ is the restriction of F on $G^{F^{mr}}$, and $M^{F^{mr}}\langle \theta^m \dot{w}_1 \rangle$ be the subgroup of $\tilde{G}^{F^{mr}}$ generated by $M^{F^{mr}}$ and $\theta^m \dot{w}_1$. Then $\delta^{(m)}$ can be extended to a character of $M^{F^{mr}}\langle \theta^m \dot{w}_1 \rangle$, which we denote by $\tilde{\delta}^{(m)}$. The restriction of $\tilde{\delta}^{(m)}$

to the coset $L^{F^{mr}}\theta\dot{w}_1\dot{w}$ may be regarded as an element of $C(L^{F^{mr}}/\sim_{F\dot{w}_1\dot{w}})$ and $a_{F^m\dot{w}_1\dot{w}}(\tilde{\delta}^{(m)}) \in C(G^{F^{mr}}/\sim_{F^m})$ is described by (5.15.1) in [S3] as follows.

$$a_{F^m\dot{w}_1\dot{w}}(\tilde{\delta}^{(m)} \mid L^{F^{mr}}\theta^m\dot{w}_1\dot{w})$$

$$= q_w^{-mr/2} q^{-l(w_1w)mr/2} \sum_{E \in (W_\delta)^\wedge_{\mathrm{ex}}} \operatorname{Tr}(T_{\gamma_1 w}, \tilde{E}(q^{mr}))(\tilde{\rho}_E \mid G^{F^{mr}}\theta^m), \tag{3.7.2}$$

where $\tilde{E}$ is an extension of E to $\tilde{W}_\delta$ and $\tilde{E}(q^{mr})$ is the irreducible representation of the algebra $\tilde{H}(q^{mr})$ (the extension of the Iwahori Hecke algebra $H(q^{mr})$ associated to the induced representation $I = \operatorname{Ind}_{P^{F^{mr}}}^{G^{F^{mr}}} \delta^{(m)}$, with basis T_w, $w \in \tilde{W}_\delta$). ρ_E is an irreducible constituent of I corresponding to E, and $\tilde{\rho}_E$ is a suitable extension to $\tilde{G}^{F^{mr}}$. Moreover q_w is a power of q associated to $w \in W_\delta$, (see 4.14, 5.15 in [S3] for the notation.)

Now by Theorem 5.13 in [S3], we can write

$$(R_0^{L_{w_1w}})^{(m)} = \mu_0^{(m)} N^{*\,-1}_{F^{mr}/F^m\dot{w}_1\dot{w}}(\tilde{\delta}^{(m)} \mid L^{F^{mr}}\theta^m\dot{w}_1\dot{w})$$

for some constant $\mu_0^{(m)} \in \bar{\mathbf{Q}}_l^*$. Moreover, we have

$$R_{xE}^{(m)} = \mu_0^{(m)} N^{*\,-1}_{F^{mr}/F^m}(\tilde{\rho}_E \mid G^{F^{mr}}\theta^m).$$

Therefore (3.7.2) implies that

$$N^{*\,-1}_{F^{mr}/F^m} \circ a_{F^m\dot{w}_1\dot{w}} \circ N^{*}_{F^{mr}/F^m\dot{w}_1\dot{w}}(\varepsilon_0^{(m)}\zeta^{m}_{A_0}\chi^{(m)}_{A_0,\varphi_0^{w_1w}})$$

$$= \sum_{E \in (W_\delta)^\wedge_{\mathrm{ex}}} \operatorname{Tr}(T_{\gamma_1 w}, \tilde{E}(q^{mr}))\, R_{xE}^{(m)}. \tag{3.7.3}$$

Now the left hand side of (3.7.3) can be rewritten by using Corollary 2.12. Then we have

$$(\varepsilon_0^{(m)}\zeta^{m}_{A_0})^{-1} \sum_{E \in (W_\delta)^\wedge_{\mathrm{ex}}} \operatorname{Tr}(T_{\gamma_1 w}, \tilde{E}(q^{mr}))\, R_{xE}^{(m)} = \sum_{A} \Big(\sum_{i \in I_A} c_i \xi_i^{m} \Big) \chi_A^{(m)} \tag{3.7.4}$$

for some $c_i, \xi_i \in \bar{\mathbf{Q}}_l^*$, where $m \in \mathscr{M}_s$, and A runs over all the elements in $\hat{G}_{\mathscr{Y}}$ such that $F^*A \simeq A$. Recall here that the Iwahori–Hecke algebra $\tilde{H}(q^{mr})$ satisfies the following orthogonality relations (cf. [C, 10.11]);

$$\sum_{x \in W_\delta} q_x^{-mr} \operatorname{Tr}(T_{\gamma_1 x}, \tilde{E}(q^{mr}))\operatorname{Tr}(T_{(\gamma_1 x)^{-1}}, \tilde{E}'(q^{mr}))$$

$$= \begin{cases} \dfrac{P_{W_\delta}(q^{mr})}{D_E(q^{mr})} \dim E & \text{if} \quad \tilde{E} \simeq \tilde{E}' \\ 0 & \text{if} \quad E \not\simeq E', \end{cases}$$

where $P_{W'_\delta}(q^{mr})$ is the Poincaré polynomial of the Coxeter group W'_δ with parameters; $P_{W'_\delta}(q^{mr}) = \sum_{w \in W'_\delta} q_w^{mr}$, and $D_E(q^{mr})$ is the generic degree of E in $H(q^{mr})$.

It is known by Benson and Curtis [BC] that the irreducible representation $E(u)$ of the generic Hecke algebra $H(u)$ is realizable on $\mathbf{Q}[u^{1/2}]$, (actually the case where W'_δ is a Weyl group of type E_8 is not covered by their result, and that case is due to Lusztig [L1, 3]). Since $\tilde{E}$ is realizable on $\mathbf{Q}$, we see that $\tilde{E}(u)$ is realizable on $\mathbf{Q}[u^{1/2}]$. In particular,

$$\mathrm{Tr}(T_{\gamma_1 w}, \tilde{E}(q^{mr})) \in \mathbf{Q}[q^{mr/2}].$$

Hence, by applying the orthogonality relations to (3.7.4), one gets

(3.7.5) For each $A \in \hat{G}_{\mathscr{Y}}$, there exist finite subsets $\{\xi_i \mid i \in I'_A\}$ and $\{c_i \mid i \in I'_A\}$ of $\bar{\mathbf{Q}}_l$ satisfying the following; for any $m \in \mathscr{M}_s$, we have

$$(\varepsilon_0^{(m)} \zeta_A^m)^{-1} P_{W'_\delta}(q^{mr}) R_{x_E}^{(m)} = \sum_A \left(\sum_{i \in I'_A} c_i \xi_i^m \right) \chi_A^{(m)},$$

where A runs over all the elements in $(\hat{G}_{\mathscr{Y}})_F$.

Here we need a variant of Dedekind's theorem.

(3.7.6) Let $\mathscr{M}_s$ be as before and let $\mathscr{M}'_s$ be a subset of $\mathscr{M}_s$ consisting of m such that $m > m_0$ for a fixed m_0. Choose $\xi_1, ..., \xi_n \in \bar{\mathbf{Q}}_l^*$ and define for each i a function f_i on $\mathscr{M}_s$ by $m \mapsto \xi_i^m$. Assume that ξ_i^s are all distinct. Then the restriction of $f_1, ..., f_n$ on $\mathscr{M}'_s$ are linearly independent.

In fact, suppose that $\sum_{i=1}^n c_i f_i = 0$ on $\mathscr{M}'_s$. If we write $m = 1 + m's$, and $m_0 = 1 + m'_0 s$, we have $\sum_{i=1}^n c_i \xi_i (\xi_i^s)^{m'} = 0$ for any $m' > m'_0$. Since ξ_i^s are all distinct, the functions $m' \mapsto (\xi_i^s)^{m'}$ on the set $\{m' \in \mathbf{N} \mid m' > m'_0\}$ are linearly independent by Dedekind's theorem. It follows that $c_i \xi_i = 0$, and so $c_i = 0$ for $i = 1, ..., n$.

We are now ready to show the lemma. It follows from (3.4.2) that there exists an integer $m_0 > 0$ such that if $m \in \mathscr{M}'_s = \{m \in \mathscr{M}_s \mid m > m_0\}$ then $R_{x_E}^{(m)}$ is equal to $\chi_{A_E}^{(m)}$ up to scalar. Then by the linearly independence of the characteristic functions of character sheaves, we have for $A \neq A_E$,

$$\sum_{i \in I'_A} c_i \xi_i^m = 0 \tag{3.7.7}$$

for any $m \in \mathscr{M}'_s$. We may assume that ξ_i^s are all distinct. Thus, by applying (3.7.6), we see that (3.7.7) holds for any $m \in \mathscr{M}_s$. Therefore we have

$$(\varepsilon_0^{(m)} \zeta_{A_0}^m)^{-1} P_{W'_\delta}(q^{mr}) R_{x_E}^{(m)} = \left(\sum_{i \in I'_{A_E}} c_i \xi_i^m \right) \chi_{A_E}^{(m)}.$$

But again by (3.4.2), we have

$$R_{x_E}^{(m)} = (-1)^{\dim \Sigma} \varepsilon_0^{(m)} \xi_{A_0}^m \chi_{A_E}^{(m)},$$

(notice that $\xi_{A_0} = \xi_{A_E}$), for any $m \in \mathscr{H}'_s$. It follows that

$$(-1)^{\dim \Sigma} P_{W_s}(q^{mr}) = \sum_{i \in I_{A_E}} c_i \xi_i^m$$

for $m \in \mathscr{H}'_s$. By applying (3.7.6), we see that this holds for any $m \in \mathscr{H}_s$. In particular, by setting $m = 1$ and using $\xi_{A_0} = \xi_{A_E}$, we have

$$R_{x_E} = (-1)^{\dim \Sigma} \varepsilon_0 \xi_{A_E} \chi_{A_E}.$$

This proves the lemma.

3.8. In view of Lemma 3.7, to prove the theorem it is enough to show (3.2.1) for a cuspidal character sheaf A_x. So, assume that $\hat{G}_{\mathscr{L}}$ contains a cuspidal character sheaf. Let $s \in T^*$ be the element corresponding to $\mathscr{L} \in \mathscr{S}(T)$. Then W_s has the same rank as W. We now assume that $s \notin (T^*)^F$, namely, $z_1 \in Z_s$ in 3.1 is not identity. Then by Lemma 4.4 in [S1], one can find an F-stable Levi subgroup L containing T and satisfying the following;

(i) L is of type $A_1 \times \cdots \times A_1$,

(ii) $z_1 \in W_L$, i.e., the class $\{s\}$ in L^* is F-stable,

(iii) s is regular semisimple in L^*.

Here we should say a bit more about the choice of L given in [S1]. Under the notation there, if G^* is of type D_n, W_s is of type $D_{m_1} \times D_{m_2}$ with γ preserving both factors, (i.e., the case (iii)), we may only consider the case where F is of split type, since w_0 is identity otherwise. Moreover, if G^* is of type D_n, W_s is of type $D_m \times D_m$ with γ permuting both factors, (i.e., the case (iv)), and if F is of split type, one can choose, as an F-stable Levi subgroup, L_J^* of type $A_1 \times \cdots \times A_1$ (m factors), corresponding to the set J of simple reflections $\{(1, 2), (3, 4), \dots, (2m-1, 2m)\}$, which is smaller than the L_J^* given there.

Now for the above choice of $L = L_J$, the set $\mathscr{E}(L^F, \{s\})$ consists of a unique representation δ_0 of L^F, which is a cuspidal representation. If we notice that $s \in (T^*)^{z_1{}^1F}$, the result in [L1, 8], in particular, (8.5.13) and (8.5.3) implies that

$$W_{\delta_0} \simeq (N_W(W_L)^{z_1{}^1F} \cap W_s^{z_1{}^1F})/(W_L \cap W_s)^{z_1{}^1F}.$$

But since s is regular semisimple in L^*, we have $W_L \cap W_s = \{1\}$. Moreover, using explicit description of z_1 and L given in [S1, 4.4], together with the preceding remark on L, we see easily that $W_s^{\gamma} = W_s^{\gamma^{-1}F} \subset N_{W}(W_L)$. It follows that we have

$$W_{\delta_0} \simeq W_s^{\gamma}. \tag{3.8.1}$$

Note that the elements in W_s^{γ} may not stabilize B_L.

Now, for a fixed $w \in W_{\delta_0}$ and its representative $\dot{w} \in N_G(L)^F$, we define an automorphism $\sigma: L \to L$ by $x \mapsto \dot{w}^{-1}x\dot{w}$. Then σ stabilizes B_L and T, and so one can consider the "twisted" character sheaves $\hat{L}^{\sigma}$, $\hat{L}^{\sigma}_{\mathcal{L}}$, etc. as in [S3, 1.3]. For each F-stable character sheaf A in $\hat{L}^{\sigma}_{\mathcal{L}}$, we fix an isomorphism $\phi_A: F^*A \xrightarrow{\sim} A$, and consider the characteristic function $\chi_A = \chi_{A,\phi_A}$, which gives rise to an L^F-invariant function on $L^F\dot{w}$.

On the other hand, δ_0 can be extended to a representation of M^F, where M is an F-stable subgroup of $N_G(L)$ generated by L and $w \in W_{\delta_0}$. Then we have

(3.8.2) There exists an extension $\tilde{\delta}_0$ of δ_0 to M^F satisfying the following. For any $w \in W_{\delta_0}$ and its representative $\dot{w} \in N_G(L)$, we have

$$\mathrm{Tr}(\dot{w}x, \tilde{\delta}_0) = \pm\chi_A(\dot{w}x) \qquad (x \in L^F)$$

for some F-stable character sheaf $A \in \hat{L}^{\sigma}_{\mathcal{L}}$, and for some choice of ϕ_A.

In fact, by using Asai's argument [A1, 2.3.2] (see also [L1, 8.8]) this is reduced to showing the similar property for an F-stable reductive group L', where L' is a direct product of some copies of GL_2, and F and σ permutes each factor. If the Frobenius map F on L is of split type, the induced map F leaves each factor of L' invariant. In this case the argument in 5.22 of [S3] can be applied to show (3.8.2). So we assume that F is of non-split type. In this case L' is written as $L' \simeq L_0 \times L_1$, where L_i are F and σ-stable, L_0 is a product of two copies of GL_2 permuted by F, and L_1 is a product of some copies of GL_2 stabilized by F. For L_1 the above argument can be applied also. So, we may consider only L_0. But in this case, by an explicit description of W_s^{γ} given in [S1, Sect. 4], (see also the preceding remark on L), we can easily verify that σ acts trivially on L_0. Hence (3.8.2) in this case follows from the corresponding result (for non-twisted character sheaves on $GL_2(\mathbf{F}_{q^2})$). Thus (3.8.2) is verified.

3.9. We shall now complete the proof of Theorem 3.2. We take $s \in T^*$ and a Levi subgroup L of G as in 3.8. Now by Corollary 4.15 in [S3], applied with $m = 1$, we have the following formula, (note that the assumption (4.12.1) in [S3] holds for L thanks to (3.8.2)).

(3.9.1) There exists $P_{A,\dot{w}}(u) \in \bar{\mathbf{Q}}_l[u, u^{-1}]$ for each $A \in \hat{G}_{\mathcal{L}}$ such that

$$\sum_{E \in W^{\wedge}_{\delta_0}} \mathrm{Tr}(T_w, E(q))\,\mathrm{Tr}(g, \rho_E) = \sum_{A \in (\hat{G}_{\mathcal{L}})_F} P_{A,\dot{w}}(q^{1/2})\,\xi_A \chi_A(g)$$

for any $g \in G^F$. If the order c of γ is equal to 1 or 2, one can take $P_{A,w}(u) \in \mathbf{Q}[u, u^{-1}]$.

We choose an irreducible representation E of W_{δ_0} such that under the isomorphism (3.8.1), E corresponds to a special representation of a unique family in $\bar{X}(W_s, \gamma)$ containing a cuspidal object. Let R_0 be the almost character of G^F corresponding to the cuspidal object $x_0 \in \bar{X}(W_s)^F$. Then we have

$$\langle \rho_E, R_0 \rangle_{G^F} \neq 0.$$

Now it follows from (3.9.1), by the orthogonality relations for the Iwahori Hecke algebra $H(q)$, that ρ_E is a $\bar{\mathbf{Q}}_l$-linear combination of $\xi_A \chi_A$, where A runs over all the F-stable character sheaves in $\hat{G}_{\mathscr{Y}}$. On the other hand, ρ_E is a $\mathbf{Q}$-linear combination of R_x for $x \in \bar{X}(W_s)^F$, and the coefficient of R_0 is non-zero. Since we have verified (3.2.1) by Lemma 3.7 for all $x \in \bar{X}(W_s)^F$ except $x = x_0$, this implies that

$$R_0 = \varepsilon_{x_0} \xi_{A_0} \chi_{A_0}$$

for some $\varepsilon_{x_0} \in \bar{\mathbf{Q}}_l$. We can take $c = 1$ or 2 unless G is of type ${}^2D_n$ or ${}^3D_4$. In this case, we see that ρ_E is a $\mathbf{Q}$-linear combination of $\xi_A \chi_A$, and so $\varepsilon_{x_0} \in \mathbf{Q}$. Since ε_{x_0} has absolute value 1, it follows that $\varepsilon_{x_0} = \pm 1$. This verifies (3.2.1) also for the case $x = x_0$, and so completes the proof of the theorem.

4. Lusztig's Conjecture for Exceptional Groups

Recall that the characteristic p of $\mathbf{F}_q$ is said to be almost good for a quasi simple group G of exceptional type, if p is good for G unless G is of type E_8, and if p is odd in the case where G is of type E_8. The aim of this section is to prove the following theorem.

Theorem 4.1. *Let G be a quasi simple group (with connected center) of exceptional type, and assume that p is almost good for G. Then for each $x \in X(W_s, \gamma)$, there exists an algebraic number ε_x of absolute value 1 such that*

$$R_x = \varepsilon_x \xi_x \chi_{A_x}. \tag{4.1.1}$$

4.2. In [S3], the theorem was proved for G of type F_4 or G_2. So, in this paper, we shall consider the remaining cases, E_6, E_7 or E_8. By induction on the rank of G, we may assume that (4.1.1) holds for any proper F-stable Levi subgroup of a parabolic subgroup of G. First we note that

(4.2.1) The statement (4.1.1) holds for $x \in X(W_s, \gamma)$ if x is not of cuspidal type.

In fact, (4.2.1) can be proved in a similar line as in the case of classical groups. But we need additional arguments in some steps. So, let L be an F-stable Levi subgroup containing T, and consider the character sheaves $\hat{L}_{\mathscr{L}}$. We assume that $\hat{L}_{\mathscr{L}}$ contains cuspidal character sheaves. If L is of classical type, the argument in §3 can be applied. So, we may assume that L is of type E_6 or E_7. Then $\hat{L}_{\mathscr{L}}$ contains two cuspidal character sheaves A_1, A_2. Let $A = \mathrm{IC}(\Sigma, \mathscr{E})[\dim \Sigma]$ be one of them and consider the associated groups $\mathscr{W}_{\mathscr{E}}$ and W_δ as in 3.4. We need to show that $W_\delta \simeq \mathscr{W}_{\mathscr{E}}$. But (5.16.1) in [S3] is not applicable since $\hat{L}_{\mathscr{L}}$ has two cuspidal character sheaves. Instead of it we argue as follows. Let $p\colon L \to L' = L/Z^0(L)$ be the natural map. Then A is expressed as $A \simeq \mathscr{E}_0 \otimes \tilde{p}A'$, where A' is a cuspidal character sheaf in $\hat{L}'_{\mathscr{L}'}$ and $\mathscr{E}_0$ is a local system on L which is the inverse image of $\mathscr{E}'_0 \in \mathscr{S}(L/L_{\mathrm{der}})$. Moreover, $\mathscr{L} \simeq (\mathscr{E}_0 \mid T) \otimes \tilde{p}\mathscr{L}'$. We note that $\mathscr{L}'$ can be chosen as $\mathscr{L}_0 = \bar{\mathbf{Q}}_l$, the constant sheaf on L'. In fact, L' contains six (resp. four) cuspidal character sheaves if L' is of type E_6 (resp. E_7), ([L3, IV, 20.4, 20.5]). Two of them are contained in $\hat{L}'_{\mathscr{L}_0}$ and others are obtained by tensoring the tame local systems on L' corresponding to the elements in the center of L'^*. Thus, by replacing $\mathscr{E}_0$ by a suitable one, if necessary, we may assume that $\mathscr{L}' = \mathscr{L}_0$.

Now take $w \in W_\delta$. By the description of W_δ in 5.16 in [S3], we have $w \in N_W(W_L)$ and $w^*\mathscr{L} \simeq \mathscr{L}$ modulo W_L. First consider the case where L is of type E_7. We may assume (up to W_L) that w stabilizes $B_{L'}$ and T' $(=p(T))$ in L'. Then $\operatorname{ad} w$ on L' is represented by $\operatorname{ad} t$ for some $t \in T'$. Thus $w^*A' \simeq A'$. Since $w^*\mathscr{L} \simeq \mathscr{L}$, we have $w^*\mathscr{E}_0 \simeq \mathscr{E}_0$. Hence $w^*A \simeq A$. Next consider the case where L is of type E_6. If ad w on L' is represented by $\operatorname{ad} t$ $(t \in T')$, the same argument may be applied to show that $w^*A \simeq A$. So, we assume that $\operatorname{ad} w$ gives rise to a non-trivial graph automorphism on L'. Then since $A' \in \hat{L}'_{\mathscr{L}_0}$, it follows from [L3, IV, Cor. 20.4] that A' is fixed by $\operatorname{ad} w$. Therefore we have $w^*A \simeq A$ for this case also. The above argument implies that $W_\delta \subset \mathscr{W}_{\mathscr{E}}$. Since the reverse inclusion is shown in 5.16 in [S3], we have $W_\delta \simeq \mathscr{W}_{\mathscr{E}}$ as required.

In the next step, we need to apply (5.20.2) in [S3] to our situation. We note that (5.20.2) is valid for this case. In fact, a cuspidal character sheaf A on L is written as $A = \mathscr{E}_0 \otimes \tilde{p}A'$ as before, with $A' \in \hat{L}'_{\mathscr{L}_0}$. Since any Frobenius map on L' stabilizes $\hat{L}'_{\mathscr{L}_0}$, and acts trivially on it, the argument in the proof of (5.20.2) works as well without change. Now (4.2.1) is proved by a similar argument as in the case of classical groups.

4.3. To prove the theorem, we consider $\hat{G}_{\mathscr{L}}$ separately for each $\mathscr{L} \in \mathscr{S}(T)^F$. By (4.2.1), we may assume that $\hat{G}_{\mathscr{L}}$ contains a cuspidal character sheaf. Then $\mathscr{L}$ corresponds to an element in the center of G^*, and $\hat{G}_{\mathscr{L}} \simeq \hat{G}_{\mathscr{L}_0}$ in a natural way (e.g. [S3, 4.6.]). Thus the proof of the theorem is reduced to the case where $\mathscr{L} = \mathscr{L}_0$. In this case, $\hat{G}_{\mathscr{L}_0}$ and $\mathscr{E}(G^F, \{1\})$ are both parametrized by $\bar{X}(W)^\gamma$, where γ is trivial or non-trivial according to the cases where F is split or non-split. In order to prove the theorem, it is enough to show the following proposition.

PROPOSITION 4.4. *Let G be a group with connected center of type E_6, E_7 or E_8. Then for any $x \in \bar{X}(W)^\gamma$, we have*

$$R_x = \varepsilon_x \xi_x \chi_{A_x}, \tag{4.4.1}$$

where $\varepsilon_x = \pm 1$, and in fact $\varepsilon_x = (-1)^{\dim G}$ if F is of split type.

4.5. The remainder of this section is devoted to the proof of the proposition. As discussed in Sect. 6 in [S3] (cf. (6.1.1)) we use the following formula due to Lusztig ([L3, III, 14.14]).

(4.5.1) Assume that F is of split type. For each $E \in W^\wedge$, let ρ_E be the corresponding irreducible character in $\mathrm{Ind}_{B^F}^{G^F} 1$. Then we have

$$\rho_E = (-1)^{\dim G} \sum_{A \in \hat{G}_{\mathscr{L}_0}} \xi_A (A : R_E^{\mathscr{L}_0}) \chi_A.$$

Note that under the parametrization $\hat{G}_{\mathscr{L}_0} \leftrightarrow \bar{X}(W)^\gamma$, we have

$$(A_x : R_E^{\mathscr{L}_0}) = \langle R_x, \rho_E \rangle_{G^F} \tag{4.5.2}$$

Now let G be as in the theorem, and we assume that F is of split type. Let L be an F-stable Levi subgroup of type D_4 containing T. Then $\hat{L}_{\mathscr{L}_0}$ contains a unique cuspidal character sheaf A_0, which is stable by F. Also $\mathscr{E}(L^F, \{1\})$ contains a unique cuspidal irreducible character δ_0. We denote by $R_{\delta_0}^L$ the corresponding almost character of L^F. Then by the similar arguments in Section 3, but using (4.5.1) instead of (3.9.1), we see that

$$R_{\delta_0}^L = (-1)^{\dim L} \xi_{A_0} \chi_{A_0}. \tag{4.5.3}$$

It follows that (4.4.1) holds for any $x = x_E \in \bar{X}(W)$ corresponding to an irreducible representation $E \in W_{\delta_0}^\wedge$. This holds also for $x = x_E$ with $E \in W^\wedge$. Furthermore, if the formula such as (4.5.3) holds for a cuspidal character δ_0' of some Levi subgroup L^F, (4.4.1) holds for any $x = x_E \in \bar{X}(W)$ for $E \in W_{\delta_0'}^\wedge$. Thus, in the following, we may only verify (4.4.1) for cuspidal objects $x \in \bar{X}(W)$.

4.6. As discussed in [S3, Sect. 6], we shall verify (4.4.1) by making use of the twisting operator t_1^*: $C(G^F/\sim) \to C(G^F/\sim)$. By Theorem 3.3 in [S3], t_1^* stabilizes the characteristic function χ_A for each character sheaf A. Its eigenvalue λ_A is determined easily by using [S3, Prop. 3.8]. It is also known by Asai [A2] and Lusztig [L1, 11.2] that t_1^* stabilizes almost character of G^F (for F: split type), i.e., $t_1^* R_x = \lambda_x R_x$ for $x \in \bar{X}(W)$, where $\lambda_x = \sigma(z)/\dim \sigma$ if $x \in \bar{X}(W)$ is given by $x = (z, \sigma) \in \mathscr{F}$ (see 5.2 in [S3]). We note that in [L1, 11.2], some modification was needed for $x = x_E$ corresponding to the exceptional characters $E \in W^\wedge$ of E_7 or E_8. But as far as the almost characters are concerned, the description of eigenvalues λ_x for R_x is given as above in a unioform way.

Now assume that G is an adjoint group of type E_6. According to [L3, IV, Cor. 20.4], the cuspidal character sheaves in $\hat{G}_{\mathscr{L}_0}$ are described as follows. First assume that $p \neq 2, 3$. Then $\hat{G}_{\mathscr{L}_0}$ contains exactly two cuspidal character sheaves A_1, A_2. They are supported by the closure of a class su in G, where s is a semisimple element such that $Z_G^0(s)$ is isogenous to $SL_3 \times SL_3 \times SL_3$, and u is a regular unipotent element in $Z_G^0(s)$. The component group $A_G(su)$ is isomorphic to $C_3 \times C_3'$, where C_3, C_3' are cyclic groups of order 3, and C_3 is generated by the image of su in $A_G(su)$. A_1 and A_2 correspond to linear characters of $A_G(su)$ which is trivial on C_3' and non-trivial on C_3. Hence $\lambda_{A_1} = \theta$, $\lambda_{A_2} = \theta^2$, (here θ is a primitive cubic root of unity). Next assume that $p = 3$. Then $\hat{G}_{\mathscr{L}_0}$ also contains two cuspidal character sheaves A_1, A_2. They are supported by the closure of a class of regular unipotent element u in G, and $A_G(u) \simeq \mathbf{Z}/3\mathbf{Z}$. A_i corresponds to a non-trivial linear character of $A_G(u)$. Thus we see that $\lambda_{A_1} = \theta$, $\lambda_{A_2} = \theta^2$ for this case also.

Now assume that F is of split type. Then G^F has exactly two almost characters R_{x_i} $(i = 1, 2)$ corresponding to cuspidal objects $x_i \in \bar{X}(W)$. We see easily that $\lambda_{x_1} = \theta$ and $\lambda_{x_2} = \theta^2$, and R_{x_i} are the unique class functions of G^F (up to scalar) having these eigenvalues. Hence R_{x_i} coincides with χ_{A_i} up tp scalar. This scalar is determined by applying (4.5.1) to the special representation $E \in W^\wedge$ of the family containing cuspidal objects $x_i \in \bar{X}(W)$. Thus (4.4.1) is verified for E_6 (of split type).

Next consider the case where G is an adjoint group of type E_7. According to [L3, IV, Prop. 20.5], the cuspidal character sheaves are described as follows. $\hat{G}_{\mathscr{L}_0}$ contains exactly two cuspidal character sheaves A_1 and A_2. They are supported by the closure of a class su in G, where s is a semisimple element such that $Z_G^0(s)$ is isogenous to $SL_4 \times SL_4 \times SL_2$, and u is a regular unipotent element of $Z_G^0(s)$. The component group $A_G(su)$ is isomorphic to $\mathbf{Z}/4\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, where $\mathbf{Z}/4\mathbf{Z}$ is generated by the image $\overline{su}$ of su in $A_G(su)$. Cuspidal character sheaves A_i correspond to linear characters ρ_i of $A_G(su)$ such that the restriction of ρ_i to $\mathbf{Z}/2\mathbf{Z}$ is trivial, and $\rho_1(\overline{su}) = \sqrt{-1}$ (resp. $\rho_2(\overline{su}) = -\sqrt{-1}$). Hence we see that $\lambda_{A_1} = \sqrt{-1}$, $\lambda_{A_2} = -\sqrt{-1}$. On the other hand, G^F has exactly two almost characters R_{x_i} corresponding to cuspidal objects $x_i \in \bar{X}(W)$. Their eigenvalues are given by $\lambda_{x_1} = \sqrt{-1}$, $\lambda_{x_2} = -\sqrt{-1}$, and R_{x_i} are the unique class functions of G^F having these eigenvalues. Therefore (4.4.1) is verified in this case in a similar way as above.

4.7. Assume now that G is a simple group of type E_8. Then $\bar{X}(W)$ contains 13 cuspidal objects, which belong to the family $\mathscr{F} = \mathscr{F}_\Gamma$ with $\Gamma \simeq S_5$. All the cuspidal objects $x \in \bar{X}(W)$ and corresponding λ_x are given as follows, (we follow the notation in [L1, 4]).

$$\begin{array}{lll} x_1 = (1, \lambda^4), & x_2 = (g_2, -\varepsilon), & x_3 = (g_3, \varepsilon\theta), \\ x_4 = (g_3, \varepsilon\theta^2), & x_5 = (g_4, \sqrt{-1}), & x_6 = (g_4, -\sqrt{-1}), \\ x_7 = (g_5, \zeta), & x_8 = (g_5, \zeta^2), & x_9 = (g_5, \zeta^3), \\ x_{10} = (g_5, \zeta^4), & x_{11} = (g_6, -\theta), & x_{12} = (g_6, -\theta^2), \\ x_{13} = (g_2', \varepsilon), & & \end{array}$$

and

$$\begin{array}{llll} \lambda_{x_1} = 1, & \lambda_{x_2} = -1, & \lambda_{x_3} = \theta, & \lambda_{x_4} = \theta^2, \\ \lambda_{x_5} = \sqrt{-1}, & \lambda_{x_6} = -\sqrt{-1}, & \lambda_{x_7} = \zeta, & \lambda_{x_8} = \zeta^2, \\ \lambda_{x_9} = \zeta^3, & \lambda_{x_{10}} = \zeta^4, & \lambda_{x_{11}} = -\theta, & \lambda_{x_{12}} = -\theta^2, \\ \lambda_{x_{13}} = 1, & & & \end{array}$$

where θ (resp. $\sqrt{-1}, \zeta$) is a primitive cubic (resp. fourth, fifth) root of unity. Next, we consider the character sheaves. $\hat{G}_{\mathscr{F}_0}$ contains 13 cuspidal character sheaves. According to [L3, IV, Prop. 21.2], they are described as follows, (we use the convention as in [S3, 6.2]).

(a) $A_1 \leftrightarrow (u, \rho)$, where $A_G(u) \simeq S_5$, and ρ is the sign character of $A_G(u)$. Hence $\lambda_{A_1} = \pm 1$.

(b) $A_2 \leftrightarrow (su, \rho)$, where $Z_G(s)$ is isogenous to $SL_2 \times E_7$ and u is a certain unipotent element in $Z_G(s)$. Since $A_G(su) \simeq \mathbf{Z}/2\mathbf{Z}$, we see that $\lambda_{A_2} = \pm 1$.

(c) $A_i \leftrightarrow (su, \rho_i)$ $(i = 3, 4)$. Here $Z_G(s)$ is isogeneous to $SL_3 \times E_6$ and u is a certain unipotent element in $Z_G(s)$. In this case, $A_G(su) \simeq \mathbf{Z}/3\mathbf{Z}$, (generated by the image $\overline{su}$ of su), and ρ_i: $\mathbf{Z}/3\mathbf{Z} \to \bar{\mathbf{Q}}_l^*$ are linear characters such that $\rho_3(\overline{su}) = \theta$, $\rho_4(\overline{su}) = \theta^2$. Hence we have $\lambda_{A_3} = \theta$, $\lambda_{A_4} = \theta^2$.

(d) $A_i \leftrightarrow (su, \rho_i)$ $(i = 5, 6)$. Here $Z_G(s)$ is isogenous to $\mathrm{Spin}_{10} \times SL_4$ and u is a certain unipotent element in $Z_G(s)$. In this case, $A_G(su) \simeq \mathbf{Z}/4\mathbf{Z}$ (generated by $\overline{su}$), and ρ_i: $\mathbf{Z}/4\mathbf{Z} \to \bar{\mathbf{Q}}_l^*$ are linear characters such that $\rho_5(\overline{su}) = \sqrt{-1}$, $\rho_6(\overline{su}) = -\sqrt{-1}$. Hence we have $\lambda_{A_5} = \sqrt{-1}$, $\lambda_{A_6} = -\sqrt{-1}$.

(e) $A_i \leftrightarrow (su, \rho_i)$ $(i=7, 8, 9, 10)$. Here $Z_G(s)$ is isogenous to $SL_5 \times SL_5$ and u is a regular unipotent element in $Z_G(s)$. In this case, $A_G(su) \simeq \mathbf{Z}/5\mathbf{Z}$ (generated by $\overline{su}$) and ρ_i: $\mathbf{Z}/5\mathbf{Z} \to \bar{\mathbf{Q}}_l^*$ are linear characters such that $\rho_i(\overline{su}) = \zeta^{i-6}$ $(i=7, 8, 9, 10)$. Hence we see that $\lambda_{A_7} = \zeta$, $\lambda_{A_8} = \zeta^2$, $\lambda_{A_9} = \zeta^3$, $\lambda_{A_{10}} = \zeta^4$.

(f) $A_i \leftrightarrow (su, \rho_i)$ $(i=11, 12)$. Here $Z_G(s)$ is isogenous to $SL_2 \times SL_3 \times SL_6$ and u is a regular unipotent element in $Z_G(s)$. In this case, $A_G(\overline{su}) \simeq \mathbf{Z}/6\mathbf{Z}$ and ρ_i: $\mathbf{Z}/6\mathbf{Z} \to \bar{\mathbf{Q}}_l$ are linear characters such that $\rho_{11}(\overline{su}) = -\theta$, $\rho_{12}(\overline{su}) = -\theta^2$. Hence we see that $\lambda_{A_{11}} = -\theta$, $\lambda_{A_{12}} = -\theta^2$.

(g) $A_{13} \leftrightarrow (su, \rho)$, where $Z_G(s)$ is isomorphic to Spin_{16} and u is a certain unipotent element in $Z_G(s)$. In this case $A_G(su) \simeq D_8$ (dihedral group of order 8), and ρ is the sign character of $A_G(su)$. Hence we see that $\lambda_{A_{13}} = \pm 1$.

We want to show all the χ_{A_i} satisfies (4.4.1). First we note that

(4.7.1) For $i=3, 4, ..., 12$, χ_{A_i} coincides with R_{x_i} up to scalar.

In fact, let ρ_{E_1} be the irreducible representation of G^F corresponding to $E_1 \leftrightarrow (1, 1) \in \mathscr{F}$. Then the function χ_{A_i} is characterized, up to scalar, as the orthogonal projection of ρ_{E_1} to the subspace of the eigenspace of t_1^* with eigenvalue λ_{A_i} spanned by the vectors orthogonal to any non-cuspidal χ_A. Since R_{x_i} satisfies this property, we get (4.7.1).

Next we show

(4.7.2) For each $i = 1, 2, 13$, R_{x_i} coincides with some χ_A (up to scalar) for a cuspidal character sheaf A.

We consider $E_2 \leftrightarrow (g_5, 1) \in \mathscr{F}$. Then we have

$$\rho_{E_2} = \tfrac{1}{5} R_{x_1} - \tfrac{1}{5} \sum_{i=9}^{12} R_{x_i} + f,$$

where f is a linear combination of R_x for non-cuspidal objects $x \in \bar{X}(W)$. Using (4.5.1), we have a similar formula,

$$\rho_{E_2} = (-1)^{\dim G} \left(\tfrac{1}{5} \xi_{x_1} \chi_{A_{x_1}} - \tfrac{1}{5} \sum_{i=9}^{12} \xi_{x_i} \chi_{A_{x_i}} \right) + f.$$

Now both formula give the orthogonal decomposition of $\rho_{E_2} - f$ as the sum of eigenvectors of t_1^*. Hence, R_{x_1} coincides with one of $\chi_{A_{x_i}}$ $(i=1, 9, ..., 12)$ up to scalar.

Next consider $E_3 \leftrightarrow (g_4, 1) \in \mathscr{F}$. Then we have

$$\rho_{E_3} = \tfrac{1}{4} R_{x_{13}} - \tfrac{1}{4} R_{x_1} + f'$$

where f' is a linear combination of R_x for non-cuspidal $x \in \bar{X}(W)$. Replacing R_{x_i} by $(-1)^{\dim G}\xi_{x_i}\chi_{A_{x_i}}$, we get the similar formula for $\chi_{A_{x_i}}$. It follows that $\{R_{x_1}, R_{x_{13}}\} = \{\chi_{A_{x_1}}, \chi_{A_{x_{13}}}\}$ up to scalar. In particular, we have $\lambda_{A_{x_1}} = 1$, $\lambda_{A_{x_{13}}} = 1$. Hence, using the classification of λ_{A_i}, we see that $\lambda_{A_{x_2}} = -1$. Now the similar argument as in (4.7.2) implies that $R_{x_2} = \chi_{A_{x_2}}$ up to scalar. This proves (4.7.2).

Now, it follows from (4.5.1) and (4.5.2) that we have

$$\langle \rho_E, (-1)^{\dim G} \xi_{A_{x_i}} \chi_{A_{x_i}} \rangle_{G^F} = \langle \rho_E, R_{x_i} \rangle_{G^F}$$

for any $E \in W^\wedge$. We note that each R_{x_i} is distinguished completely from other R_x for $x \in \bar{X}(W)$ by its eigenvalue λ_{x_i} and the above multiplicities with ρ_E. Since we know already that R_{x_i} coincides with χ_A up to scalar, this implies that $R_{x_i} = \chi_{A_{x_i}}$ up to scalar. The scalar is determined by (4.5.1). This proves (4.4.1) for the case E_8.

4.8. Finally, we consider the adjoint group of type ${}^2E_6$. Let L be an F-stable Levi subgroup of type D_4, and δ_0 a unique cuspidal irreducible character of L^F. Since F is of non-split type, (4.5.1) can not be applied. But applying the argument in 3.9 for L with $s = 1$, we have

$$R^L_{\delta_0} = \varepsilon_0 \xi_{A_0} \chi_{A_0}$$

for some $\varepsilon_{A_0} = \pm 1$. Thus, (4.4.1) is verified for any non-cuspidal object $x \in \bar{X}(W)^F$. We shall verify it for cuspidal objects x_1, x_2 in $\bar{X}(W)^F$. In [S4], the action of twisting operator $t_1^* = N^*_{F/F}$ for a non-split Frobenius map F will be duscussed. It follows from the results there, we see that R_{x_1} and R_{x_2} are the unique functions whose eigenvalue of t_1^* are θ or θ^2. (But notice that R_{x_1} and R_{x_2} are not distinguished from our result, see [S1, 3.2]). We see easily that the eigenvalues of t_1^* on χ_{A_1} and χ_{A_2} for A_1, A_2 in 4.6 are θ and θ^2, respectively. Thus, χ_{A_i} are characterized as the the projection of some ρ_E to the eigenspace with eigenvalue θ or θ^2. Hence we have $\{R_{x_1}, R_{x_2}\} = \{\chi_{A_1}, \chi_{A_2}\}$ up to scalar. This shows (4.4.1) for ${}^2E_6$. Hence Proposition 4.4 is proved, and so Theorem 4.1 follows.

5. Character Sheaves in Bad Characteristics

5.1. In this section we shall prove some results on character sheaves in the case where p is not almost good. As in [S3, Sect. 7], we denote by $\mathscr{A}(G)$ the set of admissible complexes on G. Assume now that G is a simple group of type E_8, and $p = 2$, 3 or 5. Then by [L2], $\mathscr{A}(G)$ has 13 cuspidal complexes (independent of p), which are described as follows. (For the notation on unipotent classes in G, we follow [Sp]. See 4.7 for other notation.)

I. The case $p=2$.

(a) $A_1 \leftrightarrow (u, \rho)$, where u is a unipotent element of type $2A_4$. $A_G(u) \simeq S_5$ and ρ is the sign character of S_5. Hence $\lambda_{A_1} = \pm 1$.

(b) $A_2 \leftrightarrow (u, \rho)$, where u is a unipotent element of type $E_7(a_2)+A_1$. $A_G(u) \simeq S_3 \times \mathbf{Z}/2\mathbf{Z}$ and ρ is the sign character of S_3 tensored by the non-trivial character of $\mathbf{Z}/2\mathbf{Z}$. Thus, $\lambda_{A_2} = \pm 1$.

(c) the same as (c) in 4.7.

(d) $A_i \leftrightarrow (u, \rho_i)$ $(i=5, 6)$, where u is a unipotent element of type $E_8(a_1)$. $A_G(u) \simeq \mathbf{Z}/4\mathbf{Z}$ is generated by $\bar{u}$, and $\rho_i\colon \mathbf{Z}/4\mathbf{Z} \to \bar{\mathbf{Q}}_l^*$ are linear characters such that $\rho_5(\bar{u}) = \sqrt{-1}$, $\rho_6(\bar{u}) = -\sqrt{-1}$. Thus we have $\lambda_{A_5} = \sqrt{-1}$, $\lambda_{A_6} = -\sqrt{-1}$.

(e) the same as (e) in 4.7.

(f) the same as in (f) in 4.7.

(g) $A_{13} \leftrightarrow (u, \rho)$, where u is a unipotent element of type $D_8(a_1)$. $A_G(u) \simeq D_8$, the dihedral group of order 8, and ρ is the sign character of D_8. Thus we have $\lambda_{A_{13}} = \pm 1$.

II. The case $p=3$.

(a), (b), (c), (d), (e), (g) are the same as the corresponding cases in 4.7.

(f) $A_i \leftrightarrow (u, \rho_i)$, $(i=11, 12)$, where u is a unipotent element of type E_7+A_1. $A_G(u) \simeq \mathbf{Z}/6\mathbf{Z}$, and is generated by $\bar{u}$. $\rho_i\colon \mathbf{Z}/6\mathbf{Z} \to \bar{\mathbf{Q}}_l^*$ are linear characters such that $\rho_{11}(\bar{u}) = -\theta$, $\rho_{12}(\bar{u}) = -\theta^2$. Hence we have $\lambda_{A_{11}} = -\theta$, $\lambda_{A_{12}} = -\theta^2$.

III. The case $p=5$.

(a), (b), (c), (d), (f), (g) are the same as the corresponding cases in 4.7.

(e) $A_i \leftrightarrow (u, \rho_i)$ $(i=7, 8, 9, 10)$, where u is a regular unipotent element of G. $A_G(u) \simeq \mathbf{Z}/5\mathbf{Z}$, and is generated by $\bar{u}$. $\rho_i\colon \mathbf{Z}/5\mathbf{Z} \to \bar{\mathbf{Q}}_l^*$ are linear characters such that $\rho_i(\bar{u}) = \zeta^{i-6}$. Thus we have $\lambda_{A_7} = \zeta$, $\lambda_{A_8} = \zeta^2$, $\lambda_{A_9} = \zeta^3$, $\lambda_{A_{10}} = \zeta^4$.

5.2. In the case where G is an adjoint group of type E_6 or E_7, it is known by [L3, IV, Prop. 20.3] that all the important properties such as cleanness, purity condition, etc. of character sheaves which were verified for the good prime cases hold as well for G in any characteristic. Then our result (Theorem 4.4) can be extended to those cases also without any difficulty. In the case where G is a simple group of type E_8, one can show in a similar way as before that (4.4.1) holds for each character sheaf A in $\hat{G}_{\mathscr{L}}$ subject to the condition that $\mathscr{L} \neq \mathscr{L}_0$. But for $\hat{G}_{\mathscr{L}_0}$, we have a partial result as in the case of type F_4 (see [S3, Sect. 7]) because of the lack of cleanness of $\hat{G}$. Here we only show the following result.

PROPOSITION 5.3. *Assume that G is a simple group of type E_8 and that $p=2$, 3 or 5. Then we have*

$$\mathscr{A}(G)=\hat{G}.$$

Proof. As discussed in 5.1, $\mathscr{A}(G)$ contains 13 cuspidal complexes. To prove the proposition it is enough to show that all of these cuspidal complexes are contained in $\hat{G}_{\mathscr{S}_0}$. Now, $\hat{G}_{\mathscr{S}_0}$ consists of the following five types of charactre sheaves.

(a) the character sheaves A_E corresponding to $E\in W^\wedge$,

(b) the components of $\operatorname{ind}_P^G A_0$, where P is of type D_4 and A_0 is the unique cuspidal character sheaf in $\hat{L}_{\mathscr{S}_0}$,

(c) the components of $\operatorname{ind}_P^G A_0$, $\operatorname{ind}_P^G A_0'$, where P is of type E_6 and A_0, A_0' are the cuspidal character sheaves in $\hat{L}_{\mathscr{S}_0}$,

(d) the components of $\operatorname{ind}_P^G A_0$, $\operatorname{ind}_P^G A_0'$, where P is of type E_7 and A_0, A_0' are the cuspidal character sheaves in $\hat{L}_{\mathscr{S}_0}$,

(e) cuspidal character sheaves.

In each case of (b), (c), and (d), the endomorphism algebra $\operatorname{End}_{\mathscr{M}G}\operatorname{ind}_P^G A_0$, etc. is isomorphic to the twisted group algebra of $N_G(L)/L$. In the case of (d), this endomorphism algebra is obviously the group algebra of $N_G(L)/L\simeq \mathbf{Z}/2\mathbf{Z}$. In the case of (c), $N_G(L)/L$ is isomorphic to a Coxeter group of type G_2., and one can show in a similar way as in [S3, 7.6] that $\operatorname{End}_{\mathscr{M}G}\operatorname{ind}_P^G A_0$ etc. are isomorphic to the group algebra of $N_G(L)/L$. In fact, if we consider in an appropriate $\mathbf{F}_q$-structure, the eigenvalue of χ_{A_0} (resp. $\chi_{A_0'}$) is θ (resp. θ^2), and the dimension of the space $\mathscr{V}$ spanned by the orthogonal projection of various ρ_E ($E\in W^\wedge$) to the eigenspace of t_1^* with eigenvalue θ is equal to 7. But since at most one cuspidal complex has the eigenvalue θ by 5.1, we will have $\dim\mathscr{V}\leqslant 4$ if $\operatorname{End}_{\mathscr{M}G}\operatorname{ind}_P^G A_0$ contains no one-dimensional representation. This is absurd. In particular, we see that $A_3, A_4\in\hat{G}_{\mathscr{S}_0}$.

The similar computation for the case (d) with eigenvalues $\sqrt{-1}$, $-\sqrt{-1}$ implies that $A_5, A_6\in\hat{G}_{\mathscr{S}_0}$. Now by considering the space spanned by the orthgonal projection of various ρ_E to the eigenspace with eigenvalues ζ^i (resp. $-\theta$, $-\theta^2$), we see that A_i ($i=7$, 8, 9, 10, 11, 12) are contained in $\hat{G}_{\mathscr{S}_0}$. Next consider the space $\mathscr{V}'$ spanned by the orthogonal projection of various ρ_E to the subspace of 1-eigenspace orthgonal to any R_{x_E} with $E\in W^\wedge$. Then we see that $\dim\mathscr{V}'=2$. Since $\mathscr{V}'$ is generated by characteristic functions χ_A for cuspidal character sheaves A with $\lambda_A=1$, we see by 5.1, that two of A_1, A_2, A_{13} are contained in $\hat{G}_{\mathscr{S}_0}$ and they have eigenvalue 1. It remains only one cuspidal complex A with $\lambda_A = \pm 1$, which we denote by A_*. To show that $A_* \in \hat{G}_{\mathscr{S}_0}$, we argue as follows. (Note that in the case (b). $N_G(L)/L$ is isomorphic to a Coxeter group of type F_4, and the method used in the case (c) is not enough effective to show the existence of one-dimensional representation for $\operatorname{End}_{\mathscr{M}G} \operatorname{ind}_P^G A_0$). It follows from the preceding argument, we see that χ_A coincides with some R_x up to scalar for A in (a), (c), and (d). We see also that R_x coincides with some χ_A for cuspidal objects $x = x_i$ $(i = 3, ..., 12)$ in the notation of 4.7. Moreover, the space generated by R_x for $x = x_1$ and x_{13} coincides with the space generated by χ_A for the two A among A_1, A_2 and A_{13}. Assume now that x belong to a family $\mathscr{F}'$ different from $\mathscr{F}$, and that $\lambda_x = -1$. Then it is easy to see that R_x coincides with one of χ_A for A in (b). Let $\mathscr{V}''$ be the space generated by the orthogonal projection of ρ_E (for E such that $x_E \in \mathscr{F}$) to the -1 eigenspace of t_1^*. Then it is easily verified by considering the action of t_1^* on R_x, that $\dim \mathscr{V}'' = 6$, which is equal to the number of $x \in \mathscr{F}$ such that $\lambda_x = -1$. $\mathscr{V}''$ is also generated by some characteristic functions χ_A for $A \in \hat{G}_{\mathscr{S}_0}$ such that $\lambda_A = -1$. Now suppose that $A_* \notin \hat{G}_{\mathscr{S}_0}$. Then $\mathscr{V}''$ is generated by χ_A for A in (b). Since χ_{A_*} is orthogonal to any χ_A for non-cuspidal A (cf. [L3, II, Th. 10.8]), χ_{A_*} is orthogonal to $\mathscr{V}''$. χ_{A_*} is also orthogonal to any R_x for $x \in \mathscr{F}$ with $\lambda_x \neq \pm 1$ since they belong to different eigenspaces. Moreover, χ_{A_*} is orthogonal to any almost character of G^F which belong to $\mathscr{E}(G^F, \{s\})$ with $s \neq 1$, since such almost character coincides with some χ_A up to scalar for non-cuspidal $A \in \hat{G}$. It follows that χ_{A_*} is contained in $\mathscr{V}''$. But this is absurd since cuspidal complexes are linearly independent. Thus we see that $A_* \in \hat{G}_{\mathscr{S}_0}$ and that $\lambda_{A_*} = -1$. This proves the proposition.

5.4. Finally we give a result concerning with Green functions on G. Remember that there are two types of Green functions, one is associated to the Deligne–Lusztig's virtual character $R_T^G(\theta)$ and is denoted as Q_T^G, and the other one is associated to the character sheaves on G and is denoted as $\tilde{Q}_T^G$. Both of them are functions on the set of unipotent elements of G^F. By a result of Lusztig ([L4]), these two Green functions coincide up to scalar if q is not too small. In [S3, Th. 2.2, Th. 7.4], this result was extended to arbitrary q in the case where p is almost good or G is of type F_4 or G_2. We can now state

THEOREM 5.5. *Let G be a reductive group defined over $\mathbf{F}_q$. For any p and any q, we have*

$$\tilde{Q}_T^G = (-1)^{\dim T} Q_T^G.$$

Proof. The proof is reduced to the case where G is an adjoint simple group. So, in view of the above remark, we may assume that G is an adjoint group of type E_6, E_7 or E_8 with $p = 2$, 3 or 5. By [S3, Th. 2.2], we have only to show that all the character sheaves in $\hat{G}_{\mathscr{S}_0}$ are stable by a split Frobenius map, and that the characteristic functions χ_A on G^F for $A \in \hat{G}_{\mathscr{S}_0}$ are linearly independent. But if G is an adjoint group of type E_6 or E_7, χ_A are mutually orthogonal by the cleanness of $\hat{G}$, and so they are linearly independent. Since it is easily verified from the classification of character sheaves that all A in $\hat{G}_{\mathscr{S}_0}$ are F-stable, our assertion holds for this case. We consider the case where G is a simple group of type E_8. It is verified from the classification that all the character sheaves in $\hat{G}_{\mathscr{S}_0}$ are F-stable. Moreover, it follows from Proposition 5.3 (and by the argument used there) we see that χ_A for $A \in \hat{G}_{\mathscr{S}_0}$ are linearly independent and give rise to a basis of the subspace of $C(G^F/\sim)$ generated by R_x for $x \in \bar{X}(W)$. So the theorem follows.

References

[A1] T. Asai, Endomorphism algebras of the reductive groups over $\mathbf{F}_q$ of classical type, preprint.

[A2] T. Asai, The unipotent class functions of exceptional groups over finite fields, *Comm. Algebra* **12** (1984), 2729–2857.

[A3] T. Asai, On the twisting operators on the finite classical groups, *in* "Algebraic and Topological Theories," pp. 239–282, Kinokuniya, Tokyo, 1985.

[BDD] A. A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, *Astérisque* **100** (1982).

[BC] C. T. Benson and C. W. Curtis, On the degrees and rationality of certain characters of finite Chevalley groups, *Trans. Amer. Math. Soc.* **165** (1972), 251–273.

[C] R. W. Carter, "Finite Groups of Lie Type, Conjugacy Classes and Complex Characters," Pure and Applied Mathematics, A Wiley Interscience, New York, 1985.

[L1] G. Lusztig, "Characters of Reductive Groups over a Finite Field," Annals of Mathematical Studies, Vol. 107, Princeton Univ. Press, Princeton, NJ, 1984.

[L2] G. Lusztig, Intersection cohomology complex on a reductive group, *Invent. Math.* **75** (1984), 205–272.

[L3] G. Lusztig, Character sheaves, I, *Adv. Math.* **56** (1985), 193–237; II, *Adv. Math.* **57** (1985), 226–265; III, *Adv. Math.* **57** (1985), 266–315; IV, *Adv. Math.* **59** (1986), 1–63; V, *Adv. Math.* **61** (1986), 103–155.

[L4] G. Lusztig, Green functions and character sheaves, *Ann. Math.* **131** (1990), 355–408.

[S1] T. Shoji, Shintani descent for exceptional groups over a finite field, *J. Fac. Sci. Univ. Tokyo Sect. IA* **34** (1987), 599–653.

[S2] T. Shoji, Shintani descent for algebraic groups over a finite field, I, *J. Algebra* **145** (1992), 468–524.

[S3] T. Shoji, Character sheaves and almost characters of reductive groups, *Adv. Math.* **111** (1995), 244–313.

[S4] T. Shoji, Shintani descent for algebraic groups over a finite field, II, in preparation.

[Sp] N. Spaltenstein, On the generalized Springer correspondence for exceptional groups, *in* "Algebraic Groups and Related Topics," Advanced Studies in Pure Mathematics, Vol. 6, pp. 317–338, Kinokuniya and North-Holland, Amsterdam, 1985.